DELAYS, UNSTABLE PARTICLES, AND TRANSPORT THEORY*

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Abstract

Delays associated with elementary interaction processes are investigated. The case of broad resonances is discussed in the context of reaction simulations.

1 Introduction

This study has been first motivated by problems within the transport theory for low-energy heavy-ion reactions. The common microscopic models for the reactions include the cascade and the Boltzmann-equation models. In the cascade model, a reaction is represented as a superposition of independent NN collisions. The Boltzmann-equations models account for the mean-field effects, in addition to collisions, with particles moving along curved rather than straight-line trajectories in-between the collisions. Within the models there is a flexibility in including different produced particles; the mean-field parameters may, principally, be constrained using flow data.

In simulating the reactions, it is necessary to decide how to simulate the elementary scattering processes. Some possibilities are indicated in Fig. 1. The most common prescription is the one where particles change their momenta at the distance of the closest approach. This constitutes the so-called impact scattering. Variations exist, correlating the spatial separation between the particles and the momentum transfer, as in the attractive ($qr < 0$) and repulsive ($qr > 0$) impact scattering. At low energies cross sections are isotropic and the choices include the billiard-ball scattering. Finally, the billiard-ball
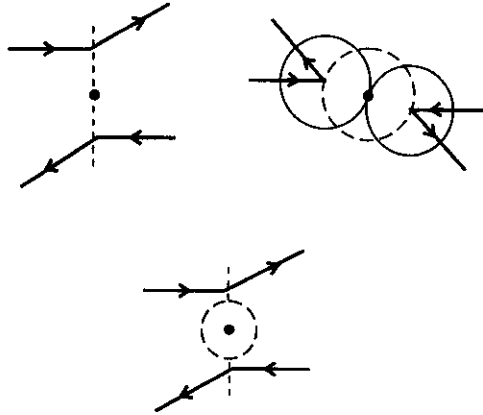

Fig. 1: Different scattering prescriptions in transport models: impact-parameter, billiard-ball, and mixed scattering.

scattering may be preferred at low impact parameters, and the impact scattering at high. This gives a so-called mixed scattering prescription.

While it is difficult to assess which scattering prescription should be favored for interactions, the results of reaction simulations exhibit surprising sensitivity to the prescriptions. Thus, in [1] variations by more than a factor of 2 in the rise of density above normal were found when using the billiard-ball and the impact-parameter scattering for nucleons in a central Ne + U reaction. In [2] the average flow angles were found to vary by nearly a factor of 2 depending whether the impact or the repulsive impact scattering was used in a U + U collision simulation. Concerning high energies, Kahana et al. found huge flow effects in the simulations of Au + Au collisions at 10.7 GeV/nucleon, when utilizing the billiard-ball scattering. Herrmann and Bertsch [4] considered an expanding pion gas in connection with the ultrarelativistic collisions and obtained large differences in source radii, such as studied through intensity interferometry, depending whether pions followed attractive or repulsive impact scattering.

These and other experiences prompted an investigation [5] to limit the short-term nature of elementary processes and establish possible links between the scattering prescriptions and the macroscopic properties of a system.

2   Wave-Packet Interactions

To assess what happens in scattering we consider an experiment where a target is located at the center of a scattering chamber of radius R, see Fig. 2. A wave-
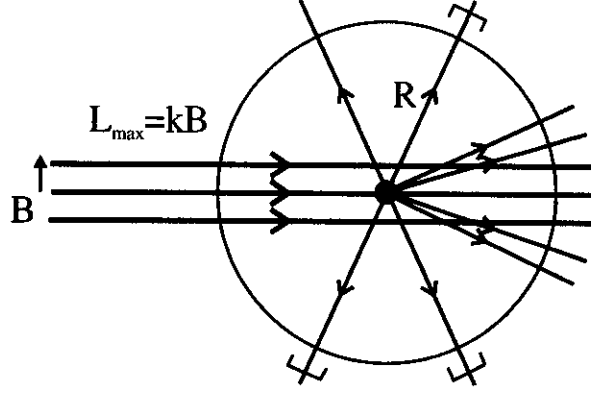
Fig. 2: Scattering chamber of radius R.

packet is directed, towards the target, asymptotically given by

$$\psi(t) = \int dE \, g(E) \left(\phi^0(\mathbf{r}, t) + f(\theta, E) \frac{e^{i(kr - Et)}}{r} \right), \tag{1}$$

where

$$\phi^0 = \frac{1}{2ikr} \sum_{\ell=0}^{L_{max}} (2\ell + 1) P_\ell(\cos\theta) \left[(-1)^{(\ell+1)} e^{-i(kr + Et)} + e^{i(kr - Et)} \right] \tag{2}$$

and

$$f(\theta, E) = \frac{1}{k} \sum_\ell (2\ell + 1) P_\ell(\cos\theta) \sin\delta_\ell \, e^{i\delta_\ell}. \tag{3}$$

The wave-packet has a transverse size B and is constructed in such a way that at the time $t = 0$ it is centered at the target.

We examine the average time spent by the packet within the chamber

$$\tau_{vol} = R^2 \frac{\int dt \, d\Omega \, \mathbf{j}(R, \Omega, t) \cdot \hat{\mathbf{r}} \, t}{N}, \tag{4}$$

and compare it to the the time $\tau_{vol}^0 = 2R/v$ spent within the chamber in the absence of interaction, $\tau_{vol} = \tau_{vol}^0 + \Delta\tau$. In (4), $\mathbf{j}$ is current density

$$\mathbf{j} = \frac{1}{2\mu i} \phi^* \overset{\leftrightarrow}{\partial} \phi, \tag{5}$$

and N is integrated incident flux through the surface of the chamber. The product $\mathbf{j} \cdot \hat{\mathbf{r}}$ in (4) is positive when a particle exits the volume and negative when a particle enters.

For a large volume there is no interference between incoming and outgoing waves

$$\tau_{vol} = \tau_{out} - \tau_{in}, \tag{6}$$

and the incoming time is found simply from (2) as $\tau_{in} = -R/v$. The outgoing waves consist of scattered and unscattered portions of the wave packet and the outgoing current has contributions from the scattered wave, unscattered wave, and the interference between the two.

To appreciate the importance of interference one may consider the difference between the net number of particles going out from and coming into the chamber

$$R^2 \int dt \, d\Omega \, \mathbf{j}(R, \Omega, t) \cdot \hat{\mathbf{r}} = N_{out} - N_{in} \, . \tag{7}$$

By integrating over the back of the chamber, the current coming in $(g(E) \simeq \delta(E))$ is found equal to

$$N_{in} = \frac{\pi \, v}{k^2} \sum_{\ell=0}^{L_{max}} (2\ell + 1) = v \, \pi B^2 \, . \tag{8}$$

The current going out consists of scattered and forward parts,

$$N_{out} = N_s + N_f \, , \tag{9}$$

and

$$N_s = \int d\Omega \, \frac{dN_s}{d\Omega} = \int d\Omega \, v|f|^2 = v\sigma \, . \tag{10}$$

If the forward part is calculated from the unscattered packet only, the current coming out in the forward direction is found to be the same as that coming in (8). Thus, the current coming out through wide angles (10) appears to be born within the chamber. The solution lies in a careful treatment of the interference between the unscattered portion of the packet and the scattered wave. In (5), the interference gives rise to terms proportional to $e^{ik(z-r)}$. For finite angles the terms vanish upon an averaging over a small macroscopic range of R but that is not the case in the forward direction. On integrating around the forward direction with the inclusion of interference term, one finds

$$N_f = \frac{\pi \, v}{k^2} \left(\sum_{\ell=0}^{L_{max}} (2\ell + 1) - 4k \operatorname{Im} f(0) \right) = v \left(\pi B^2 - \sigma \right) = N_{in} - N_s \, . \tag{11}$$

Due to $\sigma = \frac{4v}{k} \operatorname{Im} f(0)$, the interference terms precisely correct for the flux moved to finite angles.

Returning to the times, at finite angles there is no interference and the average exit time may be computed as a function of scattering angle as

$$\tau_s(\theta) = R^2 \, \frac{\int dt \, \mathbf{j_s}(R, \theta, t) \cdot \hat{\mathbf{r}} \, t}{dN_s/d\Omega} = \frac{R}{v} + \frac{1}{2i|f|^2} \left\{ f^* \, \frac{df}{dE} - f \, \frac{df^*}{dE} \right\} = \frac{R}{v} + \frac{d \operatorname{Im} \log f}{dE} \, . \tag{12}$$

The information on the delay $\Delta \tau_s$, relative to the case without interaction, is contained in the *phase* of the scattering amplitude. For scattering in only one partial wave the delay time is

$$\Delta \tau_s = \frac{d\delta_\ell}{dE}. \qquad (13)$$

Calculation of the average forward exit delay time yields

$$
\begin{aligned}
\tau_f &= \frac{R}{v} + \frac{2}{\sum_\ell^{L_{max}} (2\ell + 1) - 4k \operatorname{Im} f(0)} \frac{d}{dE}\left\{ k \operatorname{Re} f(0) \right\} \\
&\approx \frac{R}{v} + \frac{\sum_\ell^{L_{max}} (2\ell + 1)\, 2 \cos 2\delta_\ell \frac{d\delta_\ell}{dE}}{\sum_\ell^{L_{max}} (2\ell + 1)}. \qquad (14)
\end{aligned}
$$

The delay time in the forward direction involves the real part of the forward scattering amplitude.

3 Forward Delay and Mean Field

One may ask whether any effect of the delays in elementary processes is accounted for in the reaction simulations. In the impulse approximation, the real part of optical nuclear potential may be expressed in terms of the real part of the forward scattering amplitude,

$$U = n \operatorname{Re} \mathcal{T}(0) = -\frac{N}{V} \frac{2\pi v}{k} \operatorname{Re} f(0), \qquad (15)$$

indicating a possible connection to the forward delay time. Note that, when density varies in space, so does the optical potential.

To further explore the possible connection between the optical potential and the forward delay, let us consider a situation where we construct an optical potential landscape associated with some distribution of scatterers in space. One way to do it is to surround each scatterer with a well of radius R and depth $\mathcal{V}_0 = \operatorname{Re} \mathcal{T}(0)/V$, where V is the volume of a well, and to increase R until the wells merge, see Fig. 3. Let a particle be incident on a set of the scatterers at a low density. The average time spent by the incident particle within the region of scatterers will be affected by the optical potential. The time of traversal through a single well is $\tau = s/v$, where s is the length of trajectory within the well and v is group velocity. When the optical potential is put to zero, the average length of a trajectory becomes just the ratio of the volume of the well to the transverse area, $\bar{s} = V/A = \frac{4}{3}\pi R^3 / \pi R^2 = \frac{4}{3} R$. The average time
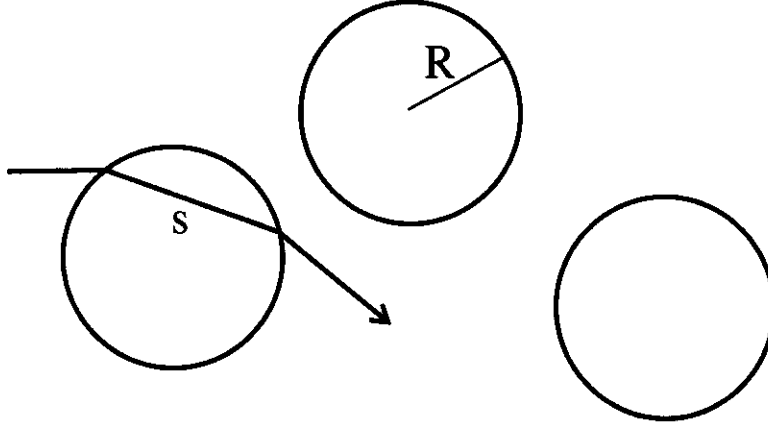
Fig. 3: Particle incident on potential wells.

within the well is $\bar{\tau} = \bar{s}/v$. When the potential is switched on, the time within the well changes compared to the case without potential, due to a change in the length of trajectory caused by refraction, and due to the change in the group velocity,

$$\Delta\bar{\tau} = \frac{\Delta\bar{s}}{v} - \frac{\bar{s}}{v^2}\Delta v = -\frac{\bar{s}}{V}\frac{1}{k^2}\frac{d}{dE}\left\{\frac{k^2}{v}\operatorname{Re}\mathcal{T}(0)\right\}$$

$$= \frac{2}{k^2R^2}\frac{d}{dE}\left\{k\operatorname{Re}f(0)\right\} \approx \frac{2}{\sum_\ell(2\ell+1)}\frac{d}{dE}\left\{k\operatorname{Re}f(0)\,.\right\} \qquad (16)$$

The above manipulation shows that the change in the average time for a single well is identical to the forward delay for a packet of transverse size equal to the size of the well, $\sum_\ell(2\ell+1) \approx k^2R^2$. Thus, the forward delay time and the mean field are, indeed, directly related.

4 Delays and Pressure

Pressure for an enclosed system may be defined as the momentum delivered to the walls per unit area and per unit time. Given a dependence of the pressure on density and temperature one can determine other thermodynamic quantities for a system. In the mean-field approximation, at low density, the correction to the ideal-gas pressure from interactions can be, conversely, obtained from the energy density. Simple manipulations allow to express the correction to the pressure in terms of the forward delay time,

$$P = P_0 + P_{mf} = nT + \frac{1}{2}n^2\operatorname{Re}\mathcal{T}(0)$$

$$= nT - \frac{1}{2} T \int dp_1 \, dp_2 \, f_1 \, f_2 \, \pi R^2 \, v \, \Delta\tau_f \,. \qquad (17)$$

Given that particles stay shorter or longer within the interaction region, compared to the free-flight time, they knock more or less often against the enclosure walls and the pressure increases or decreases, respectively, due to the interactions. Depending on the sign of the delay time the correction due to the interactions may be further interpreted in terms of excluded volume ($\Delta\tau < 0$) or a reduction in the number of degrees of freedom ($\Delta\tau > 0$).

Yet the delays associated with scattering should also affect the frequency of knocking against the walls. To the second-order in density the pressure is given completely by the Beth-Uhlenbeck formula. Manipulations of the second-order term in the formula separate out the contributions to the pressure from the forward delay time or mean field and from scattering,

$$
\begin{aligned}
P &= nT - Tn^2 \left(\frac{4\pi}{mT} \right)^{3/2} \frac{1}{2} \int \frac{dE}{2\pi} \, e^{-E/T} \sum_\ell (2\ell + 1) \, 2 \, \frac{d\delta_\ell}{dE} \\
&= P_0 + P_{mf} + P_{sc} = nT - \frac{T}{2} \int dp_1 \, dp_2 \, f_1 \, f_2 \, \pi R^2 \, v \, \Delta\tau_f \\
&\qquad - \frac{T}{2} \int dp_1 \, dp_2 \, f_1 \, f_2 \int d\Omega \, \frac{d\sigma}{d\Omega} \, v \, \Delta\tau_s \,.
\end{aligned}
\qquad (18)
$$

The scattering can affect thermodynamic properties of a system just as the mean field can. Yet, the nuclear equation of state in connection with such reaction simulations as discussed in the introduction is usually considered in terms of the mean field alone. It is apparent from (18) that, if one aims at describing solely properly the thermodynamic properties, then, to the second order in density, one can entirely absorb the effects of scattering into the forward delay or the mean field or, alternatively, the effects of the mean field into scattering. The first is accomplished by using, in place of the forward delay $\Delta\tau_f$, the time

$$\Delta\tau_f' = \Delta\tau_f + \frac{\pi R^2}{\sigma} \int d\Omega \, \frac{d\sigma}{d\Omega} \, \Delta\tau_s \,, \qquad (19)$$

or by making a replacement in the mean field

$$\text{Re}\,\mathcal{T}(0) = -\frac{\pi v}{k^2} \sum_\ell (2\ell + 1) \sin 2\delta_\ell$$

$$\Rightarrow \quad (\text{Re}\,\mathcal{T}(0))' = -\frac{\pi v}{k^2} \sum_\ell (2\ell + 1) \, 2 \, \delta_\ell \,. \qquad (20)$$

The effects of the mean field may be absorbed into scattering by using, in place of $\Delta\tau_s$, the time

$$\Delta\tau_s' = \Delta\tau_s + \frac{\sigma}{\pi R^2}\Delta\tau_f. \tag{21}$$

At a general level, the relation of thermodynamic properties to the delays may be understood in terms of ergodicity. The system spends time in a certain phase-space region, that is proportional to the density of states in that region. Changed time corresponds to the changed density of states and thus to changed thermodynamic properties.

4.1 Resonance Interactions

One of the important examples of interactions to consider is that of a Breit-Wigner resonance. The phase shift for the resonance is given by

$$\tan\delta = -\frac{\Gamma/2}{E - E_R}. \tag{22}$$

In transport simulations the resonances are given, intuitively, a lifetime $\Delta\tau_s = 1/\Gamma$. From (22) the actual scattering delay is

$$\Delta\tau_s = \frac{d\delta}{dE} = \frac{\Gamma}{2[(E - E_R)^2 + \Gamma^2/4]}. \tag{23}$$

At $E = E_R$, the scattering time is twice as large as the naively expected and it tends to zero as $|E - E_R|$ gets large, see Fig. 4. The forward delay time, normalized in such a way as if it were to be put into scattering, is

$$\Delta\tau_f \times \frac{\pi B^2}{\sigma} = \Delta\tau_s' - \Delta\tau_s = \frac{(E - E_R)^2 - \Gamma^2/4}{\Gamma[(E - E_R)^2 + \Gamma^2/4]}. \tag{24}$$

This time is negative in the vicinity of resonance. That corresponds, in particular, to the known increase in the group velocity for light near resonances in dielectrics.

The sum of the times (23) and (24) is $\Delta\tau_s' = 1/\Gamma$. Thus the naive scattering time accounts for *both* the scattering and the mean field. When employing such a time in a simulation it is inappropriate to include separately the effect of the specific resonance onto mean field. For narrow resonances, under averaging over energy, the time $\Delta\tau_s$ averages itself out to $1/\Gamma$, as the mean field averages out to zero. However, the averaging would not be justified for broad resonances.

One of the more important broad resonances in high-energy nuclear physics is the πN Δ-resonance. The width for this resonance is comparable to the
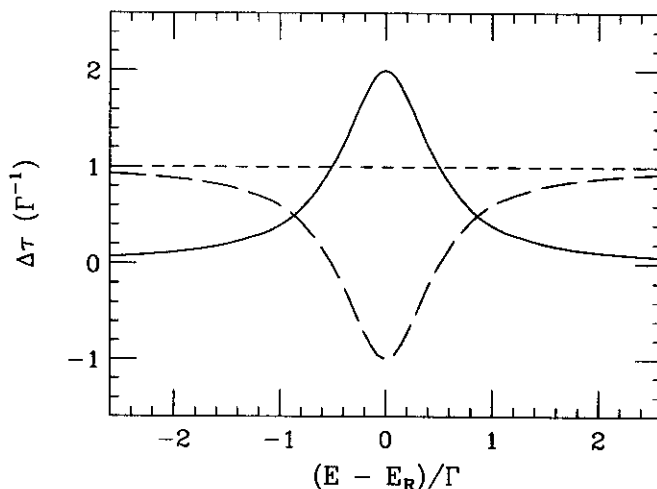
Fig. 4: Time delays in the case of a Breit-Wigner resonance as a function of energy from the resonance divided by the width. The solid line represents the time for the scattered wave $\Delta\tau_s$. The short-dashed line represents the time $\Delta\tau'_s$ which is ergodically consistent when the delay for forward wave or the mean field are neglected. Finally, the long-dashed line shows the time delay for the forward wave divided by the fraction of the incoming wave that is scattered, $\Delta\tau'_s - \Delta\tau_s$.

energy above the threshold and, as such, the width exhibits a strong energy dependence. In the immediate vicinity of the threshold, the width behaves as

$$\Gamma \propto (E - m_\pi - m_N)^{3/2}\,, \tag{25}$$

The use of the time $1/\Gamma(E)$ in simulations leads to very long-lived Δ's close to the threshold, which must be unphysical. Intuitively, the Δ's far from the center of the resonance are expected to be virtual. Figure 5 shows the delay times $\Delta\tau_s$ and $\Delta\tau'_s - \Delta\tau_s$ computed from the measured πN phase shifts. It is apparent that the scattering delay time $\Delta\tau_s$ has a quite regular behavior with energy. It is $\Delta\tau'_s - \Delta\tau_s$, within $\Delta\tau'_s \sim 1/\Gamma$, that strongly diverges close to the threshold for the Δ resonance production. It then follows that the physically unacceptable divergence of $\Delta\tau'_s$ in simulations is associated with forcing the mean field effects onto scattering when actually the scattering cross section declines, $\sigma_{\pi N \to \Delta} \propto (E - m_\pi - M_N)^{1/2}$.

With the above, the solution to the dilemma with diverging times appears straightforward. The scattering and forward delays should simply be treated separately. Yet the delays in a simulation accomplish various goals. One goal is the production of the proper overall density of states and of the corresponding proper thermodynamic properties in equilibrium. Another goal is the production of the proper number of resonances which, when decaying, may contribute
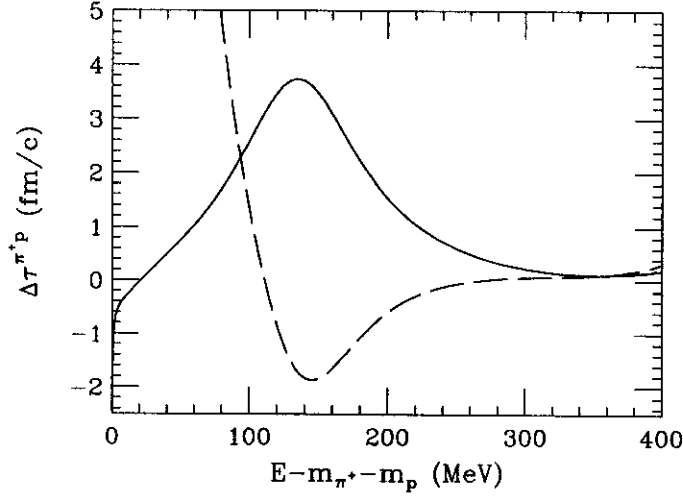
Fig. 5: Time delays for a $\pi^+ p$ system as a function of c.m. kinetic energy, computed using measured phase shifts. The solid line represents the time delay for the scattered wave averaged over angles and spin directions, $\Delta \tau_s$. The dashed line represents the forward time delay averaged over spin directions and divided by the fraction of the incoming wave that is, on the average, scattered, $\Delta \tau'_s - \Delta \tau_s$.

to the energetic photons or lepton pairs, etc. In equilibrium, the number of Δ resonances is given in terms of the resonance spectral function and the use of the time $\Delta \tau'_s = 1/\Gamma$ in scattering, during which a πN pair converts into a Δ, meets the requirement of the proper number of resonances, see below. One may expect that, when replacing $\Delta \tau'_s$ by $\Delta \tau_s$ and $\Delta \tau_f$, and when converting πN pairs into the Δ's during these times, the requirement will still be met,

$$
\begin{aligned}
n_\Delta(\mu, T) &= 16 \int d\mathbf{P}\, dm\, e^{(\mu - E)/T} \frac{\Gamma}{(m - m_\Delta)^2 + \Gamma^2/4} \\
&= 12 \int d\mathbf{P}\, dp\, e^{(\mu - E)/T}\, v\, \frac{16}{12} \frac{\pi}{p^{*2}} \frac{\Gamma^2}{(m - m_\Delta)^2 + \Gamma^2/4} \frac{1}{\Gamma} \frac{E}{m} \\
&= 12 \int d\mathbf{P}\, dp\, e^{(\mu - E)/T}\, v\, \sigma_{N\pi \to \Delta} \frac{\gamma}{\Gamma} \overset{?}{=}\ \text{scat} + \text{for}\,.
\end{aligned}
\tag{26}
$$

However, the problem which arises, is that the forward delay time for a resonance (24) is not positive definite, and one cannot simulate a negative time for the conversion of a πN pair into a Δ. A modification, relying on the possibility of moving the mean-field effects into scattering and back, can render the scattering and forward times both positive and free from any singularity,

e.g. with

$$\Delta\tau_{s\Delta} = \frac{\Gamma/4}{(m - m_\Delta)^2 + \Gamma^2/4} \qquad \Delta\tau_{f\Delta} = \frac{\sigma}{\pi B^2} \frac{(m - m_\Delta)^2}{\Gamma[(m - m_\Delta)^2 + \Gamma^2/4]} . \quad (27)$$

For the forward processes the Δ should decay back into the original pion and nucleon with their original momenta. The times (27) are consistent with the ergodic constraint, up to terms proportional to $\partial\Gamma/\partial m$ and $\partial m_\Delta/\partial m$, and with the proper number of Δ's, as related to the number of πN pairs.

5 Schemes in Simulations

The sensitivity of the reaction simulation to prescriptions in scattering, discussed in the introduction, can be understood in terms of scattering delays and changes made to the nuclear equation of state. The billiard-ball scattering gives an average delay time equal to the negative of the time within hard-core in a free passage,

$$\Delta\tau_s = -\frac{1}{v} \frac{\frac{4}{3}\pi d^3}{\pi d^2} = -\frac{4}{3} \frac{d}{v} . \quad (28)$$

The insertion of this time delay into the pressure (18) yields the repulsive excluded volume correction to the pressure, valid to within the second order in density,

$$P = nT - \frac{1}{2} n^2 T \langle \sigma v \Delta\tau_s \rangle = nT + n^2 T \frac{2\pi d^3}{3} . \quad (29)$$

The sign of the time delay and of the correction to pressure may be inverted, making the correction to pressure attractive, by replacing the hard core in the scattering by a spherical shell of radius d open in the direction of motion, (essentially replacing the convex by the concave mirror).

The repulsive impact-parameter scattering yields the average time delay and the correction to pressure nearly as large as the billiard-ball scattering,

$$\Delta\tau_s = -\frac{\pi}{3} \frac{d}{v} , \quad (30)$$

i.e. $\pi/4$ of (28). The attractive impact-scattering changes the sign in (30) and in the correction to pressure. Finally, for the impact scattering the average time-delay is zero and the correction to pressure from scattering vanishes, within the second order in density. Given *required* delay times obtained from phase shifts [5], such that $|\Delta\tau_s| < (2/3) d \langle \sin\theta \rangle = (\Delta\tau_s)_{max}$, these times may

be generated making a fraction ν of all scatterings repulsive and a fraction $1 - \nu$ attractive,

$$\nu = \frac{1}{2}\left(1 - \frac{\Delta\tau_s}{(\Delta\tau_s)_{max}}\right) . \tag{31}$$

6 Conclusions

The two- or more-body processes hardly ever take place instantaneously. In analyzing the wave-packet scattering we found that, in the consequence of interaction, both the forward and the scattered waves get delayed. The forward delays are accounted for in the mean field in reaction simulations. Both types of delay, though, through ergodicity, affect the density of states in a system and the system thermodynamic properties. To the extent that only the density of states matters, then, to the lowest order in spatial density, the same density of states and the same thermodynamic properties may be obtained by forcing all delays onto scattering or onto the forward direction. Some difficulties, though, may arise in specific situations. Thus, the unphysically long lifetimes of resonances close to threshold in simulations are specifically associated with the attempt to force the mean-field effects onto scattering.

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