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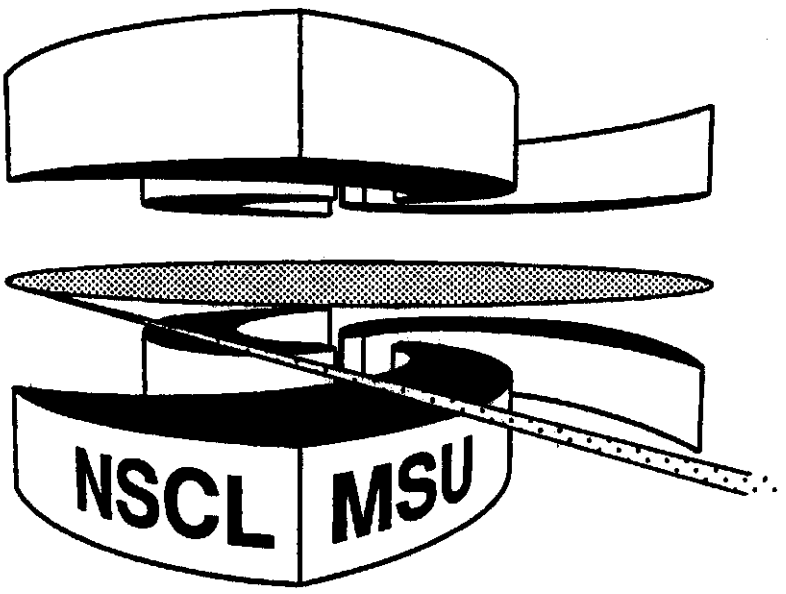

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The Differential Algebraic and Analytic Calculation of Momentum Compaction and Comparison with Tracking Codes and COSY INFINITY

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Abstract

For certain modern particle accelerators, including the planned muon collider, the accurate analysis of non-linear time-of-flight effects in the form of momentum compactions is critical for the preservation of bunch structure. A Differential Algebra-based (DA) method is presented that allows the determination of off-energy closed orbits and chromaticities to any order. Performing a coordinate transformation to the off-energy closed orbit, it is possible to compute momentum compactions analytically.

The method has been implemented in the code COSY INFINITY and is tested for two cases where analytical solutions can be obtained by hand; agreement to machine precision is found. On the other hand, comparisons are made with several codes that use conventional numerical methods for the determination of momentum compaction, and it is seen that these approaches sometimes yield rather inaccurate results, especially for higher orders.

1 Introduction

For some quasi-isochronous rings, such as the recently proposed high luminosity $2\,TeV$ muon-muon collider, it is important to keep the bunch length at minimum ($3\ mm$) in the presence of momentum spread of 0.15%. Under these conditions, the higher order momentum compaction factors can induce a large spread in the time structure (spoiling isochronicity) of every bunch of particles [4]. For realistic lattices it is impossible to calculate analytically the higher order momentum compaction factors, so one must rely on beam dynamics codes. In this paper we show that the Differential Algebraic methods, implemented in the code COSY INFINITY, give a reliable way to get not just the momentum compaction, but the chromaticities and off-energy closed orbits to any order. On the other hand, tracking codes as MAD or SINCH fail to give precise answers at higher orders due to facts that will be summarized in the last section of the paper. A comparison of results with a simple model that can be calculated analytically up to order 3 is presented.

2 The Differential Algebraic Theory

The momentum-compaction is defined as the relative orbit length variation of an off-momentum closed trajectory relative to the trajectory of the reference particle. It is obviously a function of the momentum offset $\delta = (p - p_0)/p_0$, and can be written as

$$C = C_0 \left(1 + \alpha_0 \delta + \alpha_1 \delta^2 + \alpha_2 \delta^3 + \ldots\right) \tag{1}$$

Our goal is to calculate α_i for $i = 0, 1, 2, \ldots$ This can be accomplished to any order by DA methods, and to $i = 2$ analytically for the simplified FODO cell ring described in the next section. Also, the DA method allows computation to arbitrary order of the quantities that describe how the tunes depend on energy, the so-called chromaticities. Both algorithms use the calculation of the off-energy closed orbit of a map.

The DA approach consist in the following steps. Determine the parameter-dependent fixed point of the map, followed by the linear decoupling of the planes, in case it is necessary. After these steps we obtain a map which has a block-diagonal Jacobian, each matrix element being an equivalence class, containing the value of the element and its derivatives with

respect to parameters. This form allows the computation of the parameter-dependent tune shifts and the chromaticities using a simple formula involving the trace and determinant of the matrix. Finally, to calculate the momentum compaction, a last coordinate change is necessary, in order to transform from the canonical COSY variables that measure time-of-flight, to TRANSPORT like variables that measure path length. A brief explanation of the involved steps follows.

The parameter-dependent fixed point calculation relies on the map inversion algorithm, so an outline of the algorithm is in order. All the maps of interest have no constant parts, that is they are origin preserving. In this case it is possible to compute the n-th order inverse $\mathcal{N}_n$ of a map $\mathcal{M}_n$ as long the linear part, $\mathcal{M}_1$, is invertible, which is always the case for symplectic maps. To this end, one writes $\mathcal{M}_n = \mathcal{M}_1 + \mathcal{M}_n^*$. Then we have

$$\begin{aligned} \mathcal{I}_n &= (\mathcal{M}_1 + \mathcal{M}_n^*) \circ \mathcal{N}_n = \mathcal{M}_1 \circ \mathcal{N}_n + \mathcal{M}_n^* \circ \mathcal{N}_n \Rightarrow \\ \mathcal{N}_n &= \mathcal{M}_1^{-1} \circ (\mathcal{I} - \mathcal{M}_n^* \circ \mathcal{N}_n) \end{aligned}$$

a fixed point problem for $\mathcal{N}_n$, where $\circ$ stands for composition of maps. Beginning iteration with $\mathcal{N}_n = \mathcal{I}_n$ yields convergence to the exact result in n steps because $\mathcal{M}_n^*$ is purely nonlinear.

The parameter-dependent fixed point $\vec{z}(\vec{\delta})$ is a periodic orbit of the map $\mathcal{M}_n$ satisfying $(\vec{z}(\vec{\delta}), \vec{\delta}) = \mathcal{M}_n(\vec{z}(\vec{\delta}), \vec{\delta})$. To make the map origin preserving, which in turn implies that the partial derivatives of the transfer map with respect to the parameters vanish, we perform a coordinate transformation, which in fact is a non-linear translation in parameter space. To do this, we introduce $\mathcal{I}_n^{\vec{z}}$, containing a unity map in the upper block describing the phase space variables and zeros elsewhere. Substracting $\mathcal{I}_n^{\vec{z}}$ on both parts we have $(\vec{0}, \vec{\delta}) = (\mathcal{M}_n - \mathcal{I}_n^{\vec{z}})(\vec{z}(\vec{\delta}), \vec{\delta})$ and thus,

$$(\vec{z}(\vec{\delta}), \vec{\delta}) = (\mathcal{M}_n - \mathcal{I}_n^{\vec{z}})^{-1}(\vec{0}, \vec{\delta})$$

from where we read off $\vec{z}(\vec{\delta})$ in the non-parameter lines. If energy is treated as a parameter, then $\vec{z}(\vec{\delta})$ is the off-energy closed orbit. A closer inspection reveals that the inverse exists if and only if the phase space party of $\mathcal{M}_n$ does not have 1 as an eigenvalue. However this corresponds to a fundamental resonance and is always avoided by design in accelerators.

Next, the linear decoupling of the phase planes is performed by diagonalization of the linear part of the map already expanded around the

parameter-dependent fixed point [2]. This is always possible if we require $2v$ distinct eigenvalues (in v-degrees of freedom), and this restriction is not important since otherwise the system is again on a linear resonance. Also, since we are interested in stable systems, the eigenvalues lie on the complex unit circle. Denoting the eigenvectors corresponding to eigenvalues $e^{\pm i\mu_j}$ with $s_j^{\pm}$, we obtain in the eigenvalue basis, the linear part of the map has the form

$$\begin{pmatrix} e^{i\mu_1} & & & & \\ & e^{-i\mu_1} & & 0 & \\ & & \ddots & & \\ & & & \ddots & \\ & 0 & & e^{i\mu_v} & \\ & & & & e^{-i\mu_v} \end{pmatrix}$$

We note that the eigenvectors associated with these eigenvalues are complex because the underlying matrix was real. Now perform another change of basis after which the matrix is real. For each conjugate pair of eigenvalues, we choose the real and imaginary parts of the eigenvectors as two basis vectors. The result is a matrix in block-diagonal form

$$\begin{pmatrix} [a_1]_{n-1} & [b_1]_{n-1} & & & \\ [c_1]_{n-1} & [d_1]_{n-1} & & 0 & \\ & & \ddots & & \\ & & & \ddots & \\ & 0 & & [a_v]_{n-1} & [b_v]_{n-1} \\ & & & [c_v]_{n-1} & [d_v]_{n-1} \end{pmatrix}$$

In the view of the preceding paragraphs, the matrix elements contain the derivatives with respect to parameters up to order $n-1$, that actually are the up to n-th order derivatives of the map $\mathcal{M}$. We note that all the basis changes are in fact similarity transformations, and combined with the fact that the sufficient condition for a 2x2 matrix to be symplectic is to have determinant 1, all the transformation matrices can be scaled such that the final map is still symplectic if the underlying one was symplectic. Now we can apply the simple formula involving just the trace and the determinant of the matrix [2]

$$\mu_j = \text{sign}\,(b_j) \cdot \arccos\left(\frac{T_j}{2\sqrt{D_j}}\right)$$

where T_j and D_j are the trace and determinant of the 2x2 matrices, to compute the class of μ_j

$$\left[\mu_j\right]_{n-1} = \text{sign}\left(b_j\right) \cdot \arccos\left(\frac{[a_j]_{n-1} + [b_j]_{n-1}}{2\left([a_j]_{n-1}[d_j]_{n-1} - [b_j]_{n-1}[c_j]_{n-1}\right)^{1/2}}\right)$$

Again, if energy is a parameter, then the coefficients of the relative energy deviation in the expansions of $\left[\mu_j\right]_{n-1}$ are the chromaticities.

The momentum compaction now can be readily calculated. Once we have the off-energy closed orbit, $\vec{z}(\vec{\delta})$, one last coordinate change is necessary. All DA methods being implemented in COSY INFINITY, $\vec{z}(\vec{\delta})$ is calculated using symplectic coordinates, i.e. time-of flight and conjugate momentum, the energy. The TRANSPORT-like coordinates instead are path length and momentum. Therefore, this coordinate change, performed easily in COSY [3], gives the off-momentum closed orbit, and a final step to computation of the momentum compaction is to calculate the relative change of the off-momentum with respect to the on-momentum closed orbit.

3 Simplified FODO Cell

The momentum-compaction is defined as the relative orbit length variation of an off-momentum closed trajectory relative to the trajectory of the reference orbit. It is obviously a function of the momentum dispersion factor $\delta = (p - p_0)/p_0$, and can be written as

$$C = C_0\left(1 + \alpha_0\delta + \alpha_1\delta^2 + \alpha_2\delta^3 + \ldots\right) \tag{2}$$

Our goal is to calculate analytically α_i for $i = 0, 1, 2$, and for that we choose a simplified FODO cell which is soluble analytically. See Figure 1. We take the half cell as consisting of a half focusing thin quadrupole (located at FF') followed by a homogeneous magnetic dipole and a half defocusing thin quadrupole (located at DD'). We repeat this cell until it forms a closed ring. Also, we neglect any fringe field effects. The two quadrupoles have the same integrated strength $S = KL$, where L is the quadrupole length and K is the normalized strength $K = |\ flux\ at\ pole\ tip|\ /\ (aperture * rigidity\ of\ the\ reference\ particle)$. In the thin lens approximation of the quadrupoles

the following set of limiting operations have to be taken simultaneously $Q = \{L \to 0; K \to \infty; S = KL \text{ non-zero finite}\}$.

Parameters of the off-momentum closed orbit $\left(D, D'\right)$ are the dispersion function and it's derivative with respect to s, the independent variable

$$\begin{aligned} x &= D\delta \\ x' &= D'\delta \end{aligned}$$

The dispersion function is a function of momentum and can be expanded in series

$$\begin{aligned} D &= D_0 + D_1\delta + D_2\delta^2 + \ldots \\ D' &= D'_0 + D'_1\delta + D'_2\delta^2 + \ldots \end{aligned}$$

Due to the symmetry of this cell, $D'(0) = D'(l_0) = 0$, at the entrance and exit of the half FODO cell. Also, note by $\hat{D}$ $\left(\check{D}\right)$ the value of the dispersion function at the position of the focusing (defocusing) thin quadrupoles. As the thin quadrupoles are zero length insertions, the dipoles are filling all the space and the orbit lengths of the reference particle in the half FODO cell is given by $l_0 = \rho_0\theta_0$ and the off momentum particle's by $l = \rho\theta$. In homogeneous magnetic fields the following relation holds for radius' of curvature $\rho = \rho_0(1+\delta)$.

If the ring consist of $2N$ half cells, the total orbit lengths are

$$\begin{aligned} C_0 &= 2Nl_0 \\ C &= 2Nl \end{aligned}$$

resulting the relation

$$\frac{C}{C_0} = (1+\delta)\frac{\theta}{\theta_0} \tag{3}$$

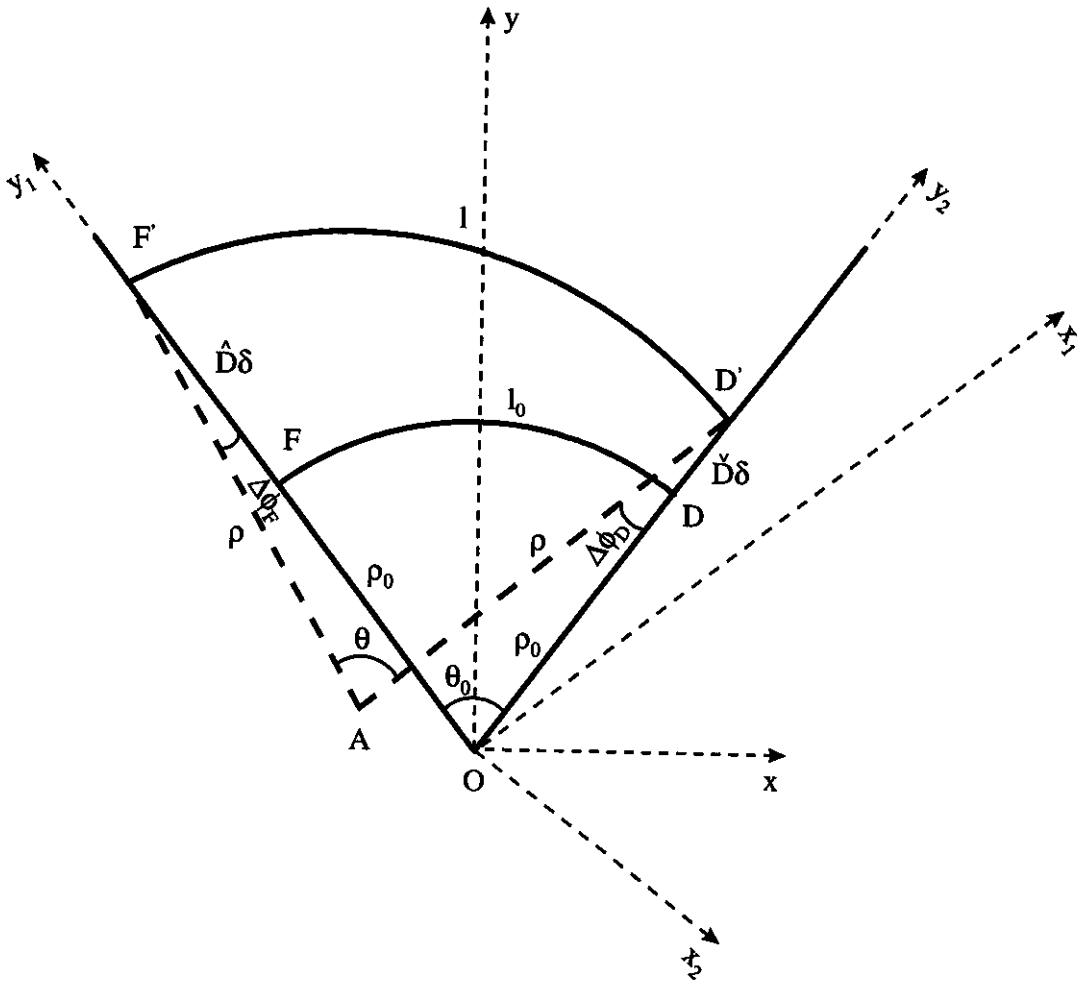

Figure 1. The outline of the half FODO cell.

So, the next necessary step is to relate θ to the dispersion functions, which in turn can be calculated following a geometric approach. From the figure it can be read off that, if we note the absolute value of the angle turned by the off-momentum closed orbit in the focusing quadrupole by $\Delta\phi_F$ and in the defocusing quadrupole by $\Delta\phi_D$, the following relation holds

$$\theta = \theta_0 + \Delta\phi_D - \Delta\phi_F \tag{4}$$

such that the total angle turned by the reference particle and the off-momentum particle in the half cell are equal.

To calculate the angles turned in the quadrupoles, we start from the general (x, a) equations of motion in a (focusing) quadrupole in particle optical coordinates.

$$\begin{aligned} x^{'} &= \frac{a}{\sqrt{(1+\delta)^2 - a^2}} \\ a^{'} &= -Kx \end{aligned}$$

On the one hand by differentiating again the first equation and plugging in in the resulting relation the second one we get

$$x^{''} = -\frac{Kx\,(1+\delta)^2}{\left[(1+\delta)^2 - a^2\right]^{3/2}}$$

and on the other hand by squaring the first equation we get

$$a^2 = (1+\delta)^2 \frac{\left(x^{'}\right)^2}{1+\left(x^{'}\right)^2}$$

Combining the last two equations and expanding up to order 3 in δ the result is

$$x^{''} =_3 -\frac{Kx}{1+\delta}\left(1+\frac{3}{2}\left(x^{'}\right)^2\right) \tag{5}$$

Integrating both sides of the equation from the start to the end of the quadrupole and taking the proper limits for the thin lens approximation, we get

$$\begin{aligned} \lim_{Q} \int_0^L x^{''} ds &= \lim_{Q} x^{'} \,|_0^L = -\tan \Delta\phi_F \\ \lim_{Q} -\int_0^L \frac{Kx}{1+\delta} ds &= -\frac{S\hat{D}\delta}{1+\delta} \end{aligned}$$

Care must be taken with the integration of the third term. Although $\lim_{Q} \int_0^L x\left(x^{'}\right)^2 ds = 0$ in the thin lens approximation, the term cannot be neglected because it is multiplied by K which in the same approximation

goes to infinity giving rise to an indeterminacy. However, it can be seen that the term is already third order in δ if instead the full D' we take just the zeroth order approximation D_0'. For this quantity it is easy to derive and solve the differential equation, which is the first order approximation of (5)

$$D_0'' + KD_0 = 0$$

with initial conditions

$$\begin{aligned} D_0(0) &= \hat{D}_0 \\ D_0'(0) &= 0 \end{aligned}$$

we get the solution

$$D_0 = \hat{D}_0 \cos\left(\sqrt{K}s\right)$$

and what we need under the integral

$$\left(D_0'\right)^2 = \hat{D}_0^2 K \sin^2\left(\sqrt{K}s\right)$$

Under this condition we can evaluate the last integral

$$\begin{aligned} \lim_Q \int_0^L KD\left(D_0'\right)^2 ds &= \lim_Q \int_0^L D\hat{D}_0^2 K^2 \sin^2\left(\sqrt{K}s\right) ds = \lim_Q \frac{D\hat{D}_0^2}{2} K^2 L \left(1 - \frac{\sin 2\sqrt{K}L}{2\sqrt{K}L}\right) \\ &= \frac{\hat{D}\hat{D}_0^2 S^2}{2} \lim_Q \frac{1}{L}\left(1 - 1 + \frac{4KL^2}{6} - \frac{16K^2L^4}{24} + \dots\right) \\ &= \frac{\hat{D}\hat{D}_0^2 S^2}{2} \lim_Q \left(\frac{2S}{3} - \frac{2S^2 L}{3} + \dots\right) = \frac{\hat{D}\hat{D}_0^2 S^3}{3} \end{aligned}$$

Gathering all the information the angle is computed from

$$\tan \Delta\phi_F =_3 \frac{S\hat{D}\delta}{1+\delta}\left(1 + \frac{S^2\hat{D}_0^2\delta^2}{2}\right)$$

by taking the arctan of the right hand side and expanding to third order in δ

$$\Delta\phi_F =_3 \hat{D}_0 S\delta - \left(\hat{D}_0 - \hat{D}_1\right) S\delta^2 + \left(\hat{D}_0 - \hat{D}_1 + \hat{D}_2 + \frac{\hat{D}_0^3 S^2}{6}\right) S\delta^3$$

Similarly, for the defocusing quadrupole

$$\Delta\phi_D =_3 \check{D}_0 S\delta - \left(\check{D}_0 - \check{D}_1\right) S\delta^2 + \left(\check{D}_0 - \check{D}_1 + \check{D}_2 + \frac{\check{D}_0^3 S^2}{6}\right) S\delta^3$$

Plugging in the above two equations in (4) and the resultant in (3) we arrive at the following

$$\frac{C}{C_0} = 1+\delta+\left(\check{D}_0 - \hat{D}_0\right) S\delta+\left(\check{D}_1 - \hat{D}_1\right) S\delta^2+\left(\check{D}_2 - \hat{D}_2 + \frac{\check{D}_0^3 S^2}{6} - \frac{\hat{D}_0^3 S^2}{6}\right) S\delta^3 \tag{6}$$

Comparing coefficients of δ in (3) and (6), we arrive to the expression for the momentum compaction factors in terms of the dispersion function's value at the quadrupoles

$$\begin{aligned} \alpha_0 &= 1 - \frac{S\left(\hat{D}_0 - \check{D}_0\right)}{\theta_0} \\ \alpha_1 &= -\frac{S\left(\hat{D}_1 - \check{D}_1\right)}{\theta_0} \\ \alpha_2 &= -\frac{S\left(\hat{D}_2 - \check{D}_2\right)}{\theta_0} - \frac{S^3\left(\hat{D}_0^3 - \check{D}_0^3\right)}{6\theta_0} \end{aligned} \tag{7}$$

4 The Geometric Solution

The off-momentum closed orbit is an arc of a circle with radius ρ and displaced center. The equation of the circle is

$$(x - x_A)^2 + (y - y_A)^2 = \rho^2$$

We express the same circle's equation in two different ways in the same coordinate system. First, take a rectangular coordinate system attached to

one edge of the dipole (the y axis), and the x axis pointing to right. In this coordinate system the center of the arc is at

$$\begin{aligned} x_{1A} &= -\rho \sin \Delta\phi_F \\ y_{1A} &= \rho_0 + \hat{D}\delta - \rho \cos \Delta\phi_F \end{aligned}$$

Thus, the equation reads

$$(x_1 + \rho \sin \Delta\phi_F)^2 + \left(y_1 - \rho_0 - \hat{D}\delta + \rho \cos \Delta\phi_F\right)^2 = \rho^2$$

Take a second coordinate system attached to the other edge of the dipole and using the same procedure, we get

$$\begin{aligned} x_{2A} &= -\rho \sin \Delta\phi_D \\ y_{2A} &= \rho_0 + \check{D}\delta - \rho \cos \Delta\phi_D \end{aligned}$$

$$(x_2 + \rho \sin \Delta\phi_D)^2 + \left(y_2 - \rho_0 - \check{D}\delta + \rho \cos \Delta\phi_D\right)^2 = \rho^2$$

Rotate the first coordinate system clockwise with $\theta_0/2$ and the second one counterclockwise with the same amount, such that the coordinate systems overlap perfectly (now with the y axis being the middle of the dipole). Hence we get the equation of the same arc in the same coordinate system

$$\begin{aligned} \left(x \cos\frac{\theta_0}{2} + y \sin\frac{\theta_0}{2} + \rho \sin \Delta\phi_F\right)^2 + \left(-x \sin\frac{\theta_0}{2} + y \cos\frac{\theta_0}{2} - \rho_0 - \hat{D}\delta + \rho \cos \Delta\phi_F\right)^2 &= \rho^2 \\ \left(x \cos\frac{\theta_0}{2} - y \sin\frac{\theta_0}{2} + \rho \sin \Delta\phi_D\right)^2 + \left(x \sin\frac{\theta_0}{2} + y \cos\frac{\theta_0}{2} - \rho_0 - \check{D}\delta + \rho \cos \Delta\phi_D\right)^2 &= \rho^2 \end{aligned}$$

Now, we can equate the coefficients in the above equation, and with the notation $t = \tan\frac{\theta_0}{2}$ the nontrivial relations are

$$\rho^2 \sin^2 \Delta\phi_F + \left(\rho_0 + \hat{D}\delta - \rho \cos \Delta\phi_F\right)^2 = \rho^2 \sin^2 \Delta\phi_D + \left(\rho_0 + \check{D}\delta - \rho \cos \Delta\phi_D\right)^2 \tag{8}$$

$$\rho\left(\sin\Delta\phi_F - \sin\Delta\phi_D\right) + t\left[2\rho_0 - \rho\left(\cos\Delta\phi_F + \cos\Delta\phi_D\right) + \delta\left(\hat{D} + \check{D}\right)\right] = 0 \tag{9}$$

$$t\rho\left(\sin\Delta\phi_F + \sin\Delta\phi_D\right) + \rho\left(\cos\Delta\phi_F - \cos\Delta\phi_D\right) + \delta\left(-\hat{D} + \check{D}\right) = 0 \tag{10}$$

Since we have only two unknowns and three equations, we use (9) and (10) to solve for the dispersion functions. We note that from (9) and (10) one can readily reproduce (8). Using the expansion relations for the dispersion functions

$$\begin{aligned}\hat{D} &= \hat{D}_0 + \hat{D}_1\delta + \hat{D}_2\delta^2 + \ldots \\ \check{D} &= \check{D}_0 + \check{D}_1\delta + \check{D}_2\delta^2 + \ldots\end{aligned}$$

and the expression for the angles $\Delta\phi_F$ and $\Delta\phi_D$ found in the previous section, we expand (9) and (10) up to third order in δ and comparing coefficients we get; at first order

$$\begin{aligned}\rho_0 S\left(\hat{D}_0 - \check{D}_0\right) + t\left(\hat{D}_0 + \check{D}_0\right) - 2\rho_0 t &= 0 \\ -\hat{D}_0 + \check{D}_0 + \rho_0 St\left(\hat{D}_0 + \check{D}_0\right) &= 0\end{aligned}$$

At second order

$$\begin{aligned}\rho_0 S\left(\hat{D}_1 - \check{D}_1\right) + t\left(\hat{D}_1 + \check{D}_1\right) + \frac{1}{2}\rho_0 S^2 t\left(\hat{D}_0^2 + \check{D}_0^2\right) &= 0 \\ -\hat{D}_1 + \check{D}_1 + \rho_0 St\left(\hat{D}_1 + \check{D}_1\right) - \frac{1}{2}\rho_0 S^2\left(\hat{D}_0^2 - \check{D}_0^2\right) &= 0\end{aligned}$$

At third order

$$\begin{aligned}\rho_0 S\left(\hat{D}_2 - \check{D}_2\right) + t\left(\hat{D}_2 + \check{D}_2\right) + \rho_0 S^2 t\left(\hat{D}_0\hat{D}_1 + \check{D}_0\check{D}_1\right) - \frac{1}{2}\rho_0 S^2 t\left(\hat{D}_0^2 + \check{D}_0^2\right) &= 0 \\ -\hat{D}_2 + \check{D}_2 + \rho_0 St\left(\hat{D}_2 + \check{D}_2\right) - \rho_0 S^2\left(\hat{D}_0\hat{D}_1 - \check{D}_0\check{D}_1\right) + \frac{1}{2}S^2\left(\hat{D}_0^2 - \check{D}_0^2\right) &= 0\end{aligned}$$

At first order the solution is simple

$$\hat{D}_0, \check{D}_0 = \frac{\rho_0 \left(1 \pm \rho_0 S t\right)}{1 + \rho_0^2 S^2}$$

or by replacing S with S/l_0 $\left(S \mapsto \frac{S}{l_0}\right)$ such that we get a dimensionless parameter for S (a value between 0 and 1)

$$\hat{D}_0, \check{D}_0 = l_0 \frac{\theta_0 \pm S t}{\theta_0^2 + S^2}$$

and introducing them in (7), we get

$$\alpha_0 = 1 - \frac{2 S^2 t}{\theta_0 \left(\theta_0^2 + S^2\right)}$$

The higher order solutions are more complicated, but a successive approximation approach implemented in Mathematica gives the solutions

$$\alpha_1 = \frac{S^4 t \left(3\theta_0^2 + S^2 t^2\right)}{\theta_0 \left(\theta_0^2 + S^2\right)^3}$$

in complete agreement with previous works. However, the correct expression for the second order momentum compaction factor is given by

$$\alpha_2 = -\frac{S^4 t}{3\theta_0 \left(\theta_0^2 + S^2\right)^5} \left[12\theta_0^6 + S^2 \left(39 + 4t^2\right) \theta_0^4 + 26 S^4 t^2 \theta_0^2 + S^6 t^2 \left(4 + 3t^2\right)\right] \tag{11}$$

In the special case $\theta_0 \ll S$ there is a valid approximation of (11) which reads up to fourth order in θ_0

$$\alpha_2 =_4 -\frac{\theta_0^2}{6} \left(1 + \frac{624 + 24S^2 + 7S^4}{16 S^4}\right)$$

5 Comparison with tracking codes, COSY and Conclusions

As pointed out in [4],[5] comparison of α_i with tracking codes such as MAD and SINCH gives agreement for $i = 0, 1$ but not $i = 2$, and the reasons

of disagreement were examined. Table 1. gives an example for $S = 0.5$ and $N = 150$.

α	SYNCH	MAD	Theory
α_0	0.00171503	0.00171503	0.0017150314
α_1	0.00267272	0.00267421	0.0026727345
α_2	0.00105371	0.00064879	-0.0000931587

Table 1. Comparison of momentum compactions up to order 3 for $N = 150$ and $S = 0.5$ among SYNCH, MAD and Theory [4]

The reasons of disagreement come from various factors related to inaccurate tracking of off-momentum particles (in the kick approximation) and numerical errors due to numerical differentiations.

In contrast, the comparison of α_2 calculated in this report and given in (11), and COSY ([6],[7],[8],[9]) results show excellent agreement, up to at least 10 digits. Moreover, the agreement is over the full range of the involved variables, that is the integrated quadrupole strengths S, and the number of the cells N. The data is gathered in Table 2. for a small ring with $N = 15$ and a large ring with $N = 150$ with the range of S being $[0, 0.9]$.

N	S	COSY	Theory	Difference
15	0.01	-.1039087182704954E-04	-.1039087110467499E-04	.7223745510592057E-12
15	0.1	-.6512543497970690E-01	-.6512543498123186E-01	.1524960713261692E-11
15	0.3	-.2492853707345641	-.2492853707353571	.7930045509141337E-12
15	0.5	-.1015105143950459	-.1015105143935272	.1518618564233520E-11
15	0.9	-.2268338355443578E-01	-.2268338355385421E-01	.5815695147681765E-12
150	0.01	-.6481116063749162E-01	-.6481116063981612E-01	.2324501702233306E-11
150	0.1	-.1030331541577536E-01	-.1030331541548904E-01	.2863178444334480E-12
150	0.3	-.2245570800125087E-03	-.2245570799716262E-03	.4088247210383422E-13
150	0.5	-.9315865569873918E-04	-.9315865603642507E-04	.3376858932454055E-12
150	0.9	-.7508271108848054E-04	-.7508271042677211E-04	.6617084267676598E-12

Table 2. Comparison of α_2 between COSY and Theory as a function of number of cells, N, and integrated and normalized quadrupole strengths, S

Finally, a short remark. There is a sign difference between the results given by COSY and the analytical calculation. However, this is consistent with COSY's choice of canonically conjugate variables, which in this case is equivalent to a replacement of α_i with $-\alpha_i$ in the momentum compaction formula (2). We can conclude that COSY gives the right result, it is a reliable code, especially when one comes to higher order effects.

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