3D Beam Dynamics in Electromagnetic rf Fields with Quadrupole Focusing and Linear Space Charge.

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Abstract

This paper describes the algorithm used in the LANA code numerical model for the beam dynamics simulation in 3D realistic electromagnetic fields [1]. The model described in [2] is upgraded in order to include a magnetic field effect in strongly asymmetric structures like quarter-wave resonators, split-ring structure or similar. A limited version of such upgrade was presented also in [3]. This effect is found to be significant for the transverse steering of light ion beams [4].

Let us consider the main harmonic of the accelerating rf field together with the space charge and external focusing fields. The equation of motion of the charged particle in the SI system of units will have a form:

$$\frac{d}{dt}(\vec{p}(\vec{x},t)) = q\cdot e\cdot \vec{E}(\vec{x})\cdot\cos(\varphi_0+\omega\cdot t) + q\cdot e\cdot[\dot{\vec{x}}\times\vec{B}(\vec{x})]\cdot\sin(\varphi_0+\omega\cdot t) + \\ + q\cdot e\cdot\vec{F}_C(\vec{x}) + q\cdot e\cdot[\dot{\vec{x}}\times\vec{B}_F(\vec{x})] \quad (1),$$

where: $\vec{p}(\vec{x},t)$ is the 3D momentum of a single particle; $q\cdot e$ – the particle's charge; $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$ are amplitudes of the standing wave rf electric and magnetic fields respectively (field profiles) in the rf cavity; φ_0 is the initial rf phase shift from the "cosine" accelerating wave (the accelerating field component has its maximum at the moment of "0" phase in the middle of the accelerating gap); ω – the rf angular frequency; $\vec{F}_C(\vec{x})$ – an "effective" coulomb force (space charge force divided by the particle charge, this value corresponds to the electrostatic repulsive field if the particle velocity would be equal to zero); $\vec{B}_F(\vec{x})$ – magnetostatic focusing field; t – the time variable; and dot in $\dot{\vec{x}}$ means $\frac{d\vec{x}}{dt}$.

The profile for the magnetic field in (1) should be given with 90° rf phase delay with respect to the electric field (the "sine" wave for the magnetic field and the "cosine" wave for the electric).

We can change the independent variable in (1) from time to rf phase (the rf time):

$$\frac{1}{c\cdot\beta_0\gamma_0}\frac{d}{d\varphi}(\gamma\cdot\dot{\vec{x}}) = \eta\cdot\vec{E}(\vec{x})\cdot\cos(\varphi_0+\varphi) + \eta\cdot\mu_0\cdot[\dot{\vec{x}}\times\vec{H}(\vec{x})]\cdot\sin(\varphi_0+\varphi) + \\ + \eta\cdot\vec{F}_C(\vec{x}) + \eta\cdot[\dot{\vec{x}}\times\vec{B}_F(\vec{x})] \quad (2)$$

where: $\eta = \frac{q\cdot e\cdot\lambda}{A\cdot 2\pi\cdot m_0c^2\cdot\beta_0\gamma_0}$; $\varphi=\omega\cdot t$; λ – rf wave length; μ_0 – magnetic permittivity of vacuum; γ – Lorentz factor; c – speed of light; $\beta_0\gamma_0$ – product of the initial relative velocity of the particle and initial Lorentz factor (the initial momentum divided by $m_0 c$); m_0 – atomic mass unit; and A – ion mass number.

For simplification of the algorithm we can assume that:

1. The values of all rf field components $E_\xi(x,y,z)$ and $H_\xi(x,y,z)$, $(\xi = x,y,z)$, are constant during the integration and are changed step-like at the end of the integration interval;

2. The external static focusing field is the ideal quadrupole field (focusing in x if g is positive and in y if g is negative respectively):
$$\begin{aligned} B_{FX}(\vec{x}) &= g \cdot y \\ B_{FY}(\vec{x}) &= g \cdot x \\ B_{FZ}(\vec{x}) &= 0 \end{aligned} \qquad (3);$$

3. The effective space charge force is linear on the integration interval and its components are:
$$\begin{aligned} F_{CX}(\vec{x}) &= Q_X \cdot (x - x_C) \\ F_{CY}(\vec{x}) &= Q_Y \cdot (y - y_C) \\ F_{CZ}(\vec{x}) &= Q_Z \cdot (\varsigma_0 + z - z_C) \end{aligned} \qquad (4);$$
Where ς_0 is an initial longitudinal displacement of the particle from the center of the charge distribution at the beginning of the current step, (x_C, y_C, z_C) are coordinates of the center of charge distribution and $Q_\xi = const$, $(\xi = x,y,z)$ along the step;

4. The dependence of the Lorentz factor $\gamma(\varphi)$ on the rf phase (time) on the integration interval is negligible:
$$\gamma(\varphi) \approx const = \frac{\gamma_0 + \gamma(\psi)}{2} \equiv \hat{\gamma} \qquad (5);$$

Taking into account simplifications 2 and 3 and the fact that $\dot{\vec{x}}$ components can be expressed as:

$$\begin{aligned} \dot{x} &= x' \cdot \beta(\varphi) \cdot c \\ \dot{y} &= y' \cdot \beta(\varphi) \cdot c \\ \dot{z} &= \beta(\varphi) \cdot c \end{aligned} \qquad (6),$$
where x' and y' are particles transverse divergences and $\beta(\varphi)$ is the particle velocity, we can rewrite (2) as:

$$\frac{d}{d\varphi}\left(\frac{\beta\gamma}{\beta_0\gamma_0}\right)=\eta\cdot E_Z(\varphi)\cdot\cos(\varphi_0+\varphi)+$$
$$+\eta\cdot\mu_0\cdot c\cdot\beta(\varphi)\cdot[x'(\varphi)\cdot H_Y(\varphi)-y'(\varphi)\cdot H_X(\varphi)]\cdot\sin(\varphi_0+\varphi)+$$
$$+\eta\cdot Q_Z(\varsigma_0+z(\varphi)-z_C(\varphi))+\eta\cdot g\cdot c\cdot\beta(\varphi)\cdot[x'(\varphi)\cdot x(\varphi)-y'(\varphi)\cdot y(\varphi)]$$
$$\frac{d}{d\varphi}\left(x'\cdot\frac{\beta\gamma}{\beta_0\gamma_0}\right)=\eta\cdot E_X(\varphi)\cdot\cos(\varphi_0+\varphi)+$$
$$+\eta\cdot\mu_0\cdot c\cdot\beta(\varphi)\cdot[y'(\varphi)\cdot H_Z(\varphi)-H_Y(\varphi)]\cdot\sin(\varphi_0+\varphi)+ \qquad (7).$$
$$+\eta\cdot Q_X(x(\varphi)-x_C(\varphi))-\eta\cdot g\cdot c\cdot\beta(\varphi)\cdot x(\varphi)$$
$$\frac{d}{d\varphi}\left(y'\cdot\frac{\beta\gamma}{\beta_0\gamma_0}\right)=\eta\cdot E_Y(\varphi)\cdot\cos(\varphi_0+\varphi)+$$
$$+\eta\cdot\mu_0\cdot c\cdot\beta(\varphi)\cdot[H_X(\varphi)-x'(\varphi)\cdot H_Z(\varphi)]\cdot\sin(\varphi_0+\varphi)+$$
$$+\eta\cdot Q_y(y(\varphi)-y_C(\varphi))+\eta\cdot g\cdot c\cdot\beta(\varphi)\cdot y(\varphi)$$

Defining $\eta'=\eta\cdot\mu_0\cdot c$ and integrating (7) on φ from 0 to ψ (ψ – rf phase (time) advance at the end of the integration interval) we can write:

$$\frac{\beta\gamma}{\beta_0\gamma_0}-1=\eta\cdot\int_0^\psi E_Z(\varphi)\cdot\cos(\varphi_0+\varphi)\cdot d\varphi+$$
$$+\eta'\cdot\int_0^\psi\beta(\varphi)\cdot[x'(\varphi)\cdot H_Y(\varphi)-y'(\varphi)\cdot H_X(\varphi)]\cdot\sin(\varphi_0+\varphi)\cdot d\varphi+$$
$$+\eta\cdot Q_Z\cdot\left(\varsigma_0\cdot\psi+\int_0^\psi(z(\varphi)-z_C(\varphi))\cdot d\varphi\right)+$$
$$+\eta\cdot g\cdot c\cdot\int_0^\psi\beta(\varphi)\cdot[x'(\varphi)\cdot x(\varphi)-y'(\varphi)\cdot y(\varphi)]\cdot d\varphi$$
$$x'\cdot\frac{\beta\gamma}{\beta_0\gamma_0}-x'_0=\eta\cdot\int_0^\psi E_X(\varphi)\cdot\cos(\varphi_0+\varphi)\cdot d\varphi+$$
$$+\eta'\cdot\int_0^\psi\beta(\varphi)\cdot[y'(\varphi)\cdot H_Z(\varphi)-H_Y(\varphi)]\cdot\sin(\varphi_0+\varphi)\cdot d\varphi+$$
$$+\eta\cdot Q_X\cdot\int_0^\psi(x(\varphi)-x_C(\varphi))\cdot d\varphi-\eta\cdot g\cdot c\cdot\int_0^\psi\beta(\varphi)\cdot x(\varphi)\cdot d\varphi$$
$$y'\cdot\frac{\beta\gamma}{\beta_0\gamma_0}-y'_0=\eta\cdot\int_0^\psi E_Y(\varphi)\cdot\cos(\varphi_0+\varphi)\cdot d\varphi+$$
$$+\eta'\cdot\int_0^\psi\beta(\varphi)\cdot[H_X(\varphi)-x'(\varphi)\cdot H_Z(\varphi)]\cdot\sin(\varphi_0+\varphi)\cdot d\varphi+ \qquad (8),$$
$$+\eta\cdot Q_Y\cdot\int_0^\psi(y(\varphi)-y_C(\varphi))\cdot d\varphi+\eta\cdot g\cdot c\cdot\int_0^\psi\beta(\varphi)\cdot y(\varphi)\cdot d\varphi$$

The right-hand sides, functions of ψ, can be defined as corresponding scaled relative momentum changes: $P_Z(\psi)$, $P_X(\psi)$ and $P_Y(\psi)$ respectively.

We can solve system (8) by means of the iterative procedure [5] that is described below.

We can rewrite the first equation in (8_1) like:

$$\beta\gamma = \beta_0\gamma_0 \cdot (1 + P_Z(\psi)) \quad (9),$$

or:

$$\beta(\psi) = \frac{\beta_0\gamma_0 \cdot (1 + P_Z(\psi))}{\sqrt{1 + (\beta_0\gamma_0)^2 (1 + P_Z(\psi))^2}} = \beta_0 \cdot \frac{(1 + P_Z(\psi))}{\sqrt{1 + \beta_0^2 \cdot P_Z(\psi)(2 + P_Z(\psi))}} \quad (10).$$

This equation (10) can be decomposed into the power series with respect to the $P_Z(\psi)$ as a small parameter. Keeping the terms up to $P_Z(\psi)^2$ in these series we can write:

$$\beta(\psi) \approx \beta_0 \cdot \left(1 + \frac{1}{\gamma_0^2} \cdot P_Z(\psi) - \frac{3}{2} \cdot \frac{\beta_0^2}{\gamma_0^2} \cdot P_Z(\psi)^2 \right) \quad (11).$$

In order to find the phase (time) advance on the integration step ψ we should find first the final longitudinal coordinate $z(\psi)$ and equate it to the length of the longitudinal integration step – L. From (6_3) follows:

$$z(\psi) = c \cdot \int \beta(t)dt = \frac{\lambda}{2\pi} \int_0^{\psi} \beta(\varphi)d\varphi = \nu \cdot \int_0^{\psi} \frac{\beta(\varphi)}{\beta_0} d\varphi = L \quad (12),$$

where: $\nu = \frac{\beta_0\lambda}{2\pi}$. Here we can define:

$$\psi_0 = \frac{2\pi}{\beta_0\lambda} \cdot L = \frac{1}{\nu} \cdot L \quad (13).$$

When we put L from (12) in (13), we get:

$$\psi_0 = \frac{1}{\nu} \cdot \nu \cdot \int_0^{\psi} \frac{\beta(\varphi)}{\beta_0} d\varphi = \int_0^{\psi} \left(1 + \frac{\beta(\varphi)}{\beta_0} - 1 \right) d\varphi = \psi + \int_0^{\psi} \left(\frac{\beta(\varphi)}{\beta_0} - 1 \right) d\varphi \quad (14),$$

and respectively:

$$\psi = \psi_0 - \int_0^{\psi} \left(\frac{\beta(\varphi)}{\beta_0} - 1 \right) d\varphi \quad (15),$$

and applying (11):

$$\psi = \psi_0 - \frac{1}{\gamma_0^2} \cdot \int_0^{\psi} P_Z(\varphi)d\varphi + \frac{3}{2} \cdot \frac{\beta_0^2}{\gamma_0^2} \cdot \int_0^{\psi} P_Z(\varphi)^2 d\varphi \quad (16).$$

This equation can be iterated in order to find the rf phase (time) advance on the integration step ψ. But before doing so let us finish derivation of the transverse coordinate formulas. Using (9) we can express second and third equations in (8) like:

$$x' = \frac{\beta_0 \gamma_0}{\beta\gamma} \cdot (x'_0 + P_X(\psi)) = \frac{(x'_0 + P_X(\psi))}{(1 + P_Z(\psi))}$$
$$y' = \frac{\beta_0 \gamma_0}{\beta\gamma} \cdot (y'_0 + P_Y(\psi)) = \frac{(y'_0 + P_Y(\psi))}{(1 + P_Z(\psi))} \qquad (17).$$

In order to find x and y coordinates we could rewrite (17) as:

$$\dot{x} \cdot \frac{1}{\beta \cdot c} = \frac{1}{\beta \cdot c} \cdot \frac{2\pi \cdot c}{\lambda} \cdot \frac{dx}{d\varphi} = \frac{\beta_0 \gamma_0}{\beta\gamma} \cdot (x'_0 + P_X(\psi)) \qquad (18),$$

and hence:

$$\frac{dx}{d\varphi} = \nu \cdot \frac{\gamma_0}{\gamma} \cdot (x'_0 + P_X(\psi)) \qquad (19).$$

After the integration we get (expressions for y are similar):

$$x(\psi) = x_0 + \nu \cdot \left(\int_0^{\psi} \frac{\gamma_0}{\gamma(\varphi)} d\varphi + \int_0^{\psi} P_X(\varphi) \frac{\gamma_0}{\gamma(\varphi)} d\varphi \right) \qquad (20).$$

Applying the simplification 4 from page 2 and defining $\nu' = \nu \cdot \frac{\gamma_0}{\hat{\gamma}}$ we can write:

$$x(\psi) = x_0 + \nu' \cdot \left(x'_0 \cdot \psi + \int_0^{\psi} P_X(\varphi) d\varphi \right)$$
$$y(\psi) = y_0 + \nu' \cdot \left(y'_0 \cdot \psi + \int_0^{\psi} P_Y(\varphi) d\varphi \right) \qquad (21).$$

In order to present $E_\xi(\varphi)$ and $H_\xi(\varphi)$ in the form suitable for integration in the right-hand side of (8) we can decompose the field components amplitudes into the Taylor series at the moment ψ after the beginning of the integration step. This decomposition is:

$$E_\xi(\varphi) = E_{0\xi} + \left. \frac{dE_\xi}{d\varphi} \right|_{\psi} \cdot (\varphi|_{\psi} - \psi)$$
$$H_\xi(\varphi) = H_{0\xi} + \left. \frac{dH_\xi}{d\varphi} \right|_{\psi} \cdot (\varphi|_{\psi} - \psi) \qquad (22).$$

According to the simplification 1 from page 2:

$$\left. \frac{dE_\xi}{d\varphi} \right|_{\psi} = \Delta E_\xi \cdot \delta(\varphi - \psi)$$
$$\left. \frac{dH_\xi}{d\varphi} \right|_{\psi} = \Delta H_\xi \cdot \delta(\varphi - \psi) \qquad (23).$$

Hence we can write:

$$E_\xi(\varphi) = E_{0\xi} + \Delta E_\xi \cdot \delta(\varphi - \psi) \cdot \left(\int_0^\varphi \frac{\beta(\tilde{\varphi})}{\beta_0} d\tilde{\varphi} - \psi\right)$$
$$H_\xi(\varphi) = H_{0\xi} + \Delta H_\xi \cdot \delta(\varphi - \psi) \cdot \left(\int_0^\varphi \frac{\beta(\tilde{\varphi})}{\beta_0} d\tilde{\varphi} - \psi\right) \quad (24),$$

where $\varphi|_\psi$ is presented according to (12) as $\varphi|_\psi = \frac{z(\psi)}{v}$.

Now we can start the iteration to define $\beta(\psi)$ and $\psi(\psi)$ that can be used in the right-hand side of the equations (8) and in (9), (16), (17), (21) respectively.

The "0" iteration corresponds to the absent external rf fields, hence $\Delta E_\xi = \Delta H_\xi = 0$ and $\beta = \beta_0$, so from (8) we have:

$$P_{0\xi}(\psi) = A_\xi \cdot [\sin(\varphi_0 + \psi) - \sin(\varphi_0)] + B_\xi \cdot [\cos(\varphi_0 + \psi) - \cos(\varphi_0)] + C_\xi \cdot \psi + D_\xi \cdot \frac{\psi^2}{2} \quad (25),$$

where:

$$A_\xi = \eta \cdot E_{0\xi}$$

$$B_Z = \eta' \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot [y_0' \cdot H_{0X} - x_0' \cdot H_{0Y}]$$
$$B_X = \eta' \cdot \beta_0 \cdot \left[H_{0Y} - y_0' \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot H_{0Z}|\right]$$
$$B_Y = \eta' \cdot \beta_0 \cdot \left[x_0' \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot H_{0Z} - H_{0X}|\right]$$

$$C_Z = \eta \cdot \left[Q_Z \cdot \varsigma_0 + g \cdot c \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot [x_0' \cdot x_0 - y_0' \cdot y_0]\right]$$
$$C_X = \eta \cdot [(Q_X - g \cdot c \cdot \beta_0) \cdot x_0 - Q_X \cdot x_{0C}]$$
$$C_Y = \eta \cdot [(Q_Y + g \cdot c \cdot \beta_0) \cdot y_0 - Q_Y \cdot y_{0C}]$$

$$D_Z = \eta \cdot \left[Q_Z \cdot v \cdot \left(1 - \frac{\beta_C}{\beta_0}\right) + g \cdot c \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot v' \cdot [x_0'^2 - y_0'^2]\right]$$
$$D_X = \eta \cdot v' \cdot [(Q_X - g \cdot c \cdot \beta_0) \cdot x_0' - Q_X \cdot x_{0C}']$$
$$D_Y = \eta \cdot v' \cdot [(Q_Y + g \cdot c \cdot \beta_0) \cdot y_0' - Q_Y \cdot y_{0C}'] \quad (26).$$

Applying (25) for (11) we can get:

$$\beta_1(\psi) = \beta_0 \cdot \left(1 + \frac{1}{\gamma_0^2} \cdot P_{0Z}(\psi) - \frac{3}{2} \cdot \frac{\beta_0^2}{\gamma_0^2} \cdot P_{0Z}(\psi)^2\right) \quad (27),$$

and using (27) in (12) we can get:

$$\varphi|_{\psi} = \frac{z_1(\psi)}{v} = \int_0^{\psi} \left(1 + \frac{1}{\gamma_0^2} \cdot P_{0Z}(\varphi) - \frac{3}{2} \cdot \frac{\beta_0^2}{\gamma_0^2} \cdot P_{0Z}(\varphi)^2 \right) d\varphi$$
$$= \psi + \frac{1}{\gamma_0^2} \cdot \int_0^{\psi} P_{0Z}(\varphi) d\varphi - \frac{3}{2} \cdot \frac{\beta_0^2}{\gamma_0^2} \cdot \int_0^{\psi} P_{0Z}(\varphi)^2 d\varphi \qquad (28).$$

Defining:

$$U_{\xi}(\psi) \equiv \int_0^{\psi} P_{0\xi}(\varphi) d\varphi$$
$$R(\psi) \equiv \int_0^{\psi} P_{0Z}(\varphi)^2 d\varphi \qquad (29),$$

we can write:

$$(\varphi|_{\psi} - \psi) = \frac{1}{\gamma_0^2} \cdot U_Z(\varphi) - \frac{3}{2} \cdot \frac{\beta_0^2}{\gamma_0^2} \cdot R(\varphi) \qquad (30).$$

When we apply (30) for (24) we can see that the term with $R(\varphi)$ in (29), which is proportional to $O(P_{0Z}(\psi)^2)$, gets multiplied by ΔE_{ξ} (or ΔH_{ξ}), which is of the same order as E_Z – the field amplitude itself, and because $P_{0Z}(\psi) \sim E_Z$ as well, we can say that the term $R(\varphi) \cdot \Delta E_{\xi} \sim O(P_{0Z}(\psi)^3)$ and is of the third order with respect to the relative change of the longitudinal momentum. To stay consistent with the decomposition (11) we should neglect this term in (24) as well. Following this argumentation we can rewrite (24) using (30) for the first iteration as:

$$E_{1\xi}(\varphi) = E_{0\xi} + \frac{1}{\gamma_0^2} \cdot \Delta E_{\xi} \cdot U_Z(\varphi) \cdot \delta(\varphi - \psi)$$
$$H_{1\xi}(\varphi) = H_{0\xi} + \frac{1}{\gamma_0^2} \cdot \Delta H_{\xi} \cdot U_Z(\varphi) \cdot \delta(\varphi - \psi) \qquad (31).$$

Now we can use (31) and (27) for the next and final iteration of (8):

$$
\begin{aligned}
P_{1Z}(\psi) = {} & P_{0Z}(\psi) + \eta \cdot \frac{1}{\gamma_0^2} \cdot \Delta E_Z \cdot U_Z(\psi) \cdot \cos(\varphi_0 + \psi) + \\
& + \eta' \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot \left[H_{0Y} \cdot V_X(\psi) - H_{0X} \cdot V_Y(\psi) \right] + \\
& + \eta' \cdot \frac{\beta_0}{\gamma_0 \cdot \gamma_1} \cdot \left[x_0' \cdot \Delta H_{0Y} - y_0' \cdot \Delta H_{0X} \right] \cdot U_Z(\psi) \cdot \sin(\varphi_0 + \psi) + \\
& + \eta \cdot \nu \cdot \frac{1}{\gamma_0^2} \cdot Q_Z \cdot S_Z(\psi) + \eta \cdot g \cdot c \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot \left[x_0 \cdot U_X(\psi) - y_0 \cdot U_Y(\psi) \right] + \\
& + \eta \cdot \nu' \cdot g \cdot c \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot \left[x_0' \cdot \left(S_X(\psi) + G_X(\psi) \right) - y_0' \cdot \left(S_Y(\psi) + G_Y(\psi) \right) \right] \\
P_{1X}(\psi) = {} & P_{0X}(\psi) + \eta \cdot \frac{1}{\gamma_0^2} \cdot \Delta E_{0X} \cdot U_Z(\psi) \cdot \cos(\varphi_0 + \psi) + \\
& + \eta' \cdot \beta_0 \cdot \left[\frac{\gamma_0}{\hat{\gamma}} \cdot H_{0Z} \cdot V_Y(\psi) - \frac{1}{\gamma_0^2} \cdot H_{0Y} \cdot V_Z(\psi) \right] + \\
& + \eta' \cdot \beta_0 \cdot \left[\frac{1}{\gamma_0 \cdot \gamma_1} \cdot y_0' \cdot \Delta H_{0Z} - \frac{1}{\gamma_0^2} \cdot \Delta H_{0Y} \right] \cdot U_Z(\psi) \cdot \sin(\varphi_0 + \psi) + \\
& + \eta \cdot \nu' \cdot \left(Q_X - g \cdot c \cdot \beta_0 \right) \cdot S_X(\psi) - \\
& - \eta \cdot g \cdot c \cdot \beta_0 \cdot \frac{1}{\gamma_0^2} \cdot \left[x_0 \cdot U_Z(\psi) + \nu' \cdot x_0' \cdot G_Z(\psi) \right] \\
P_{1Y}(\psi) = {} & P_{0Y}(\psi) + \eta \cdot \frac{1}{\gamma_0^2} \cdot \Delta E_{0Y} \cdot U_Z(\psi) \cdot \cos(\varphi_0 + \psi) + \\
& + \eta' \cdot \beta_0 \cdot \left[\frac{1}{\gamma_0^2} \cdot H_{0X} \cdot V_Z(\psi) - \frac{\gamma_0}{\hat{\gamma}} \cdot H_{0Z} \cdot V_X(\psi) \right] + \\
& + \eta' \cdot \beta_0 \cdot \left[\frac{1}{\gamma_0^2} \cdot \Delta H_{0X} - \frac{1}{\gamma_0 \cdot \gamma_1} \cdot x_0' \cdot \Delta H_{0Z} \right] \cdot U_Z(\psi) \cdot \sin(\varphi_0 + \psi) + \\
& + \eta \cdot \nu' \cdot \left(Q_Y + g \cdot c \cdot \beta_0 \right) \cdot S_Y(\psi) + \\
& + \eta \cdot g \cdot c \cdot \beta_0 \cdot \frac{1}{\gamma_0^2} \cdot \left[y_0 \cdot U_Z(\psi) + \nu' \cdot y_0' \cdot G_Z(\psi) \right]
\end{aligned}
\qquad (32),
$$

where: $\gamma_1 = \gamma(\psi)$ is the Lorentz factor at the end of the step, $U_Z(\psi)$ defined in (29), and:

$$
\begin{aligned}
V_\xi(\psi) &\equiv \int_0^\psi P_{0\xi}(\varphi) \cdot \sin(\varphi_0 + \varphi) d\varphi \\
S_\xi(\psi) &\equiv \int_0^\psi U_\xi(\varphi) d\varphi \\
G_\xi(\psi) &\equiv \int_0^\psi \varphi \cdot P_{0\xi}(\varphi) d\varphi
\end{aligned}
\qquad (33).
$$

For the final formulae (16) and (21) we need the integrals of the functions (32):

$$\int_0^{\psi} P_{1Z}(\varphi)d\varphi = U_Z(\psi) + \eta \cdot \frac{1}{\gamma_0^2} \cdot \Delta E_Z \cdot Wc(\psi) +$$
$$+\eta' \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot [H_{0Y} \cdot M_X(\psi) - H_{0X} \cdot M_Y(\psi)] +$$
$$+\eta' \cdot \frac{\beta_0}{\gamma_0 \cdot \gamma_1} \cdot [x_0' \cdot \Delta H_{0Y} - y_0' \cdot \Delta H_{0X}] \cdot Ws(\psi) +$$
$$+\eta \cdot \nu \cdot \frac{1}{\gamma_0^2} \cdot Q_Z \cdot T_Z(\psi) + \eta \cdot g \cdot c \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot [x_0 \cdot S_X(\psi) - y_0 \cdot S_Y(\psi)] +$$
$$+\eta \cdot \nu' \cdot g \cdot c \cdot \beta_0 \cdot \frac{\gamma_0}{\hat{\gamma}} \cdot [x_0' \cdot (T_X(\psi) + J_X(\psi)) - y_0' \cdot (T_Y(\psi) + J_Y(\psi))]$$

$$\int_0^{\psi} P_{1X}(\varphi)d\varphi = U_X(\psi) + \eta \cdot \frac{1}{\gamma_0^2} \cdot \Delta E_{0X} \cdot Wc(\psi) +$$
$$+\eta' \cdot \beta_0 \cdot \left[\frac{\gamma_0}{\hat{\gamma}} \cdot H_{0Z} \cdot M_Y(\psi) - \frac{1}{\gamma_0^2} \cdot H_{0Y} \cdot M_Z(\psi)\right] +$$
$$+\eta' \cdot \beta_0 \cdot \left[\frac{1}{\gamma_0 \cdot \gamma_1} \cdot y_0' \cdot \Delta H_{0Z} - \frac{1}{\gamma_0^2} \cdot \Delta H_{0Y}\right] \cdot Ws(\psi) +$$
$$+\eta \cdot \nu' \cdot (Q_X - g \cdot c \cdot \beta_0) \cdot T_X(\psi) -$$
$$-\eta \cdot g \cdot c \cdot \beta_0 \cdot \frac{1}{\gamma_0^2} \cdot [x_0 \cdot S_Z(\psi) + \nu' \cdot x_0' \cdot J_Z(\psi)]$$

$$\int_0^{\psi} P_{1Y}(\varphi)d\varphi = U_Y(\psi) + \eta \cdot \frac{1}{\gamma_0^2} \cdot \Delta E_{0Y} \cdot Wc(\psi) +$$
$$+\eta' \cdot \beta_0 \cdot \left[\frac{1}{\gamma_0^2} \cdot H_{0X} \cdot M_Z(\psi) - \frac{\gamma_0}{\hat{\gamma}} \cdot H_{0Z} \cdot M_X(\psi)\right] + \qquad (34),$$
$$+\eta' \cdot \beta_0 \cdot \left[\frac{1}{\gamma_0^2} \cdot \Delta H_{0X} - \frac{1}{\gamma_0 \cdot \gamma_1} \cdot x_0' \cdot \Delta H_{0Z}\right] \cdot Ws(\psi) +$$
$$+\eta \cdot \nu' \cdot (Q_Y + g \cdot c \cdot \beta_0) \cdot T_Y(\psi) +$$
$$+\eta \cdot g \cdot c \cdot \beta_0 \cdot \frac{1}{\gamma_0^2} \cdot [y_0 \cdot S_Z(\psi) + \nu' \cdot y_0' \cdot J_Z(\psi)]$$

where:

$$
\begin{aligned}
&Ws(\psi) \equiv \int_0^{\psi} U_Z(\varphi)\cdot\sin(\varphi_0+\varphi)d\varphi \\
&Wc(\psi) \equiv \int_0^{\psi} U_Z(\varphi)\cdot\cos(\varphi_0+\varphi)d\varphi \\
&M_\xi(\psi) \equiv \int_0^{\psi} V_\xi(\varphi)d\varphi \\
&T_\xi(\psi) \equiv \int_0^{\psi} S_\xi(\varphi)d\varphi \\
&J_\xi(\psi) \equiv \int_0^{\psi} G_\xi(\varphi)d\varphi
\end{aligned} \qquad (35).
$$

Also for (16) we need $\int_0^{\psi} P_{1Z}(\varphi)^2 d\varphi$. All terms in ($32_1$) except for the first one are of the order of $O(P_{0Z}(\psi)^2)$, so in the square of (32_1) all terms except $P_{0Z}(\varphi)^2$ are of the order higher then two. Hence using the same argumentation as for (31) we can write:

$$
\int_0^{\psi} P_{1Z}(\varphi)^2 d\varphi \approx \int_0^{\psi} P_{0Z}(\varphi)^2 d\varphi = R(\psi) \qquad (36).
$$

The upper integration limit and the argument of the corresponding functions must be consistent with the first iteration (27) and (30) and its value can be expressed as:

$$
\psi_1(\psi_0) = \psi_0 - \frac{1}{\gamma_0^2}\cdot U_Z(\psi_0) + \frac{3}{2}\cdot\frac{\beta_0^2}{\gamma_0^2}\cdot R(\psi_0) \qquad (37),
$$

Now we can rewrite (16), (9), (21) and (17) in terms of $P_{1\xi}(\psi)$ and its integrals:

$$
\begin{aligned}
&\psi(\psi_1) = \psi_0 - \frac{1}{\gamma_0^2}\cdot\int_0^{\psi_1} P_{1Z}(\varphi)d\varphi + \frac{3}{2}\cdot\frac{\beta_0^2}{\gamma_0^2}\cdot\int_0^{\psi_1} P_{1Z}(\varphi)^2 d\varphi \\
&\beta(\psi_1) = \frac{\beta_0\gamma_0\cdot(1+P_{1Z}(\psi_1))}{\sqrt{1+(\beta_0\gamma_0)^2(1+P_{1Z}(\psi_1))^2}} \\
&x(\psi_1) = x_0 + v'\cdot\left(x_0'\cdot\psi_1 + \int_0^{\psi_1} P_{1X}(\varphi)d\varphi\right) \\
&y(\psi_1) = y_0 + v'\cdot\left(y_0'\cdot\psi_1 + \int_0^{\psi_1} P_{1Y}(\varphi)d\varphi\right) \\
&x'(\psi_1) = \frac{\beta_0\gamma_0}{\beta\gamma}\cdot(x_0' + P_{1X}(\psi_1)) = \frac{(x_0' + P_{1X}(\psi_1))}{(1+P_{1Z}(\psi_1))} \\
&y'(\psi_1) = \frac{\beta_0\gamma_0}{\beta\gamma}\cdot(y_0' + P_{1Y}(\psi_1)) = \frac{(y_0' + P_{1Y}(\psi_1))}{(1+P_{1Z}(\psi_1))}
\end{aligned} \qquad (38).
$$

These equations express the final phase space coordinates of the particle as transcendental functions of the initial coordinates using (32), (34) and (36) in terms of the functions (29), (33) and (35) with argument for these functions defined by (37). Detailed representation of these functions is given in the Appendix A.

These formulae are successfully applied in the numerical model of the LANA code for the simulation of beam dynamics in the realistic 3D fields produced by means of the codes like MAFIA, HFSS or similar. Figure 1 shows the simulation results obtained with LANA code using the same fields as in the publication [4]. The curves show exactly the same characteristic behavior though the results of the LANA simulation predict a slightly bigger (~10%) magnetic field effect. This agreement is acceptable taking into account the difference in the modes used in the LANA code and in the calculations from [4].

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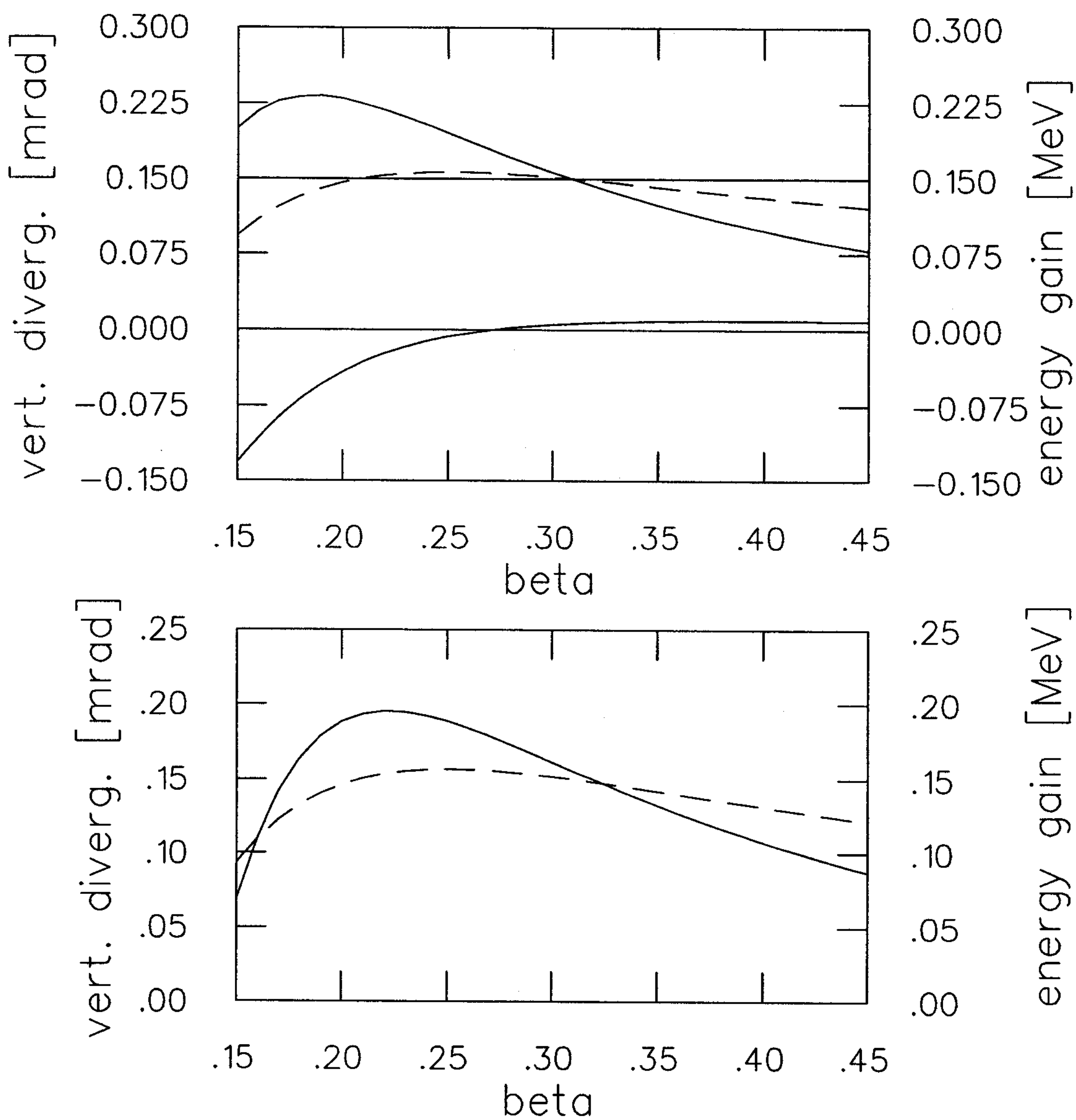

Figure 1. Partial vertical deflection contribution from electric (lower solid curve) and magnetic (upper solid curve) field together with the energy gain (dashed curve) -- top graph; and total deflection (solid curve) with the same energy gain (dashed curve) -- bottom graph.

APPENDIX A
Mathematica 4.1 evaluation of the formulae for
"3D Beam Dynamics in Electromagnetic RF Fields with Quadrupole Focusing and Linear Space Charge", D. Gorelov, NSCL, MSU, January, 2002

Pz[ψ_] := Az (Sin[φ0 + ψ] - Sin[φ0]) + Bz (Cos[φ0 + ψ] - Cos[φ0]) + Cz ψ + Dz ψ^2 / 2

Uz[ψ_] = Collect[Simplify[$\int_0^{\psi}$ Pz[x] dx], {Az, Bz, Cz, Dz}]

$$\frac{Cz\,\psi^2}{2}+\frac{Dz\,\psi^3}{6}+Az\,(\mathrm{Cos}[\phi0]-\mathrm{Cos}[\phi0+\psi]-\psi\,\mathrm{Sin}[\phi0])+Bz\,(-\psi\,\mathrm{Cos}[\phi0]-\mathrm{Sin}[\phi0]+\mathrm{Sin}[\phi0+\psi])$$

R[ψ_] = Collect[Simplify[$\int_0^{\psi}$ ComplexExpand[Pz[x]^2] dx], {Az, Bz, Cz, Dz}]

$$\begin{aligned}&\frac{Cz^2\,\psi^3}{3}+\frac{1}{4}\,Cz\,Dz\,\psi^4+\frac{Dz^2\,\psi^5}{20}+\\&Az\left(Bz\left(-\frac{3}{2}\,\mathrm{Cos}[2\,\phi0]-\frac{1}{2}\,\mathrm{Cos}[2\,(\phi0+\psi)]+2\,\mathrm{Cos}[2\,\phi0+\psi]+\psi\,\mathrm{Sin}[2\,\phi0]\right)+\right.\\&\quad Cz\,(-2\,\psi\,\mathrm{Cos}[\phi0+\psi]-2\,\mathrm{Sin}[\phi0]-\psi^2\,\mathrm{Sin}[\phi0]+2\,\mathrm{Sin}[\phi0+\psi])+\\&\quad\left.Dz\left(-2\,\mathrm{Cos}[\phi0]+2\,\mathrm{Cos}[\phi0+\psi]-\psi^2\,\mathrm{Cos}[\phi0+\psi]-\frac{1}{3}\,\psi^3\,\mathrm{Sin}[\phi0]+2\,\psi\,\mathrm{Sin}[\phi0+\psi]\right)\right)+\\&Bz\left(Cz\,((-2-\psi^2)\,\mathrm{Cos}[\phi0]+2\,\mathrm{Cos}[\phi0+\psi]+2\,\psi\,\mathrm{Sin}[\phi0+\psi])+\right.\\&\quad\left.Dz\left(-\frac{1}{3}\,\psi^3\,\mathrm{Cos}[\phi0]+2\,\psi\,\mathrm{Cos}[\phi0+\psi]+2\,\mathrm{Sin}[\phi0]-2\,\mathrm{Sin}[\phi0+\psi]+\psi^2\,\mathrm{Sin}[\phi0+\psi]\right)\right)+\\&Bz^2\left(\psi+\frac{1}{2}\,\psi\,\mathrm{Cos}[2\,\phi0]+\frac{3}{4}\,\mathrm{Sin}[2\,\phi0]-\mathrm{Sin}[\psi]+\frac{1}{4}\,\mathrm{Sin}[2\,(\phi0+\psi)]-\mathrm{Sin}[2\,\phi0+\psi]\right)+\\&Az^2\left(\psi-\frac{1}{2}\,\psi\,\mathrm{Cos}[2\,\phi0]-\frac{3}{4}\,\mathrm{Sin}[2\,\phi0]-\mathrm{Sin}[\psi]-\frac{1}{4}\,\mathrm{Sin}[2\,(\phi0+\psi)]+\mathrm{Sin}[2\,\phi0+\psi]\right)\end{aligned}$$

Wc[ψ_] = Collect[Simplify[$\int_0^{\psi}$ Uz[x] Cos[φ0 + x] dx], {Az, Bz, Cz, Dz}]

$$\begin{aligned}&\frac{1}{12}\,Cz\,(12\,\psi\,\mathrm{Cos}[\phi0+\psi]+12\,\mathrm{Sin}[\phi0]-12\,\mathrm{Sin}[\phi0+\psi]+6\,\psi^2\,\mathrm{Sin}[\phi0+\psi])+\\&\frac{1}{12}\,Dz\,(12\,\mathrm{Cos}[\phi0]-12\,\mathrm{Cos}[\phi0+\psi]+6\,\psi^2\,\mathrm{Cos}[\phi0+\psi]-12\,\psi\,\mathrm{Sin}[\phi0+\psi]+2\,\psi^3\,\mathrm{Sin}[\phi0+\psi])+\\&\frac{1}{12}\,Az\,(-6\,\psi-6\,\psi\,\mathrm{Cos}[\psi]+6\,\psi\,\mathrm{Cos}[2\,\phi0+\psi]+3\,\mathrm{Sin}[2\,\phi0]+12\,\mathrm{Sin}[\psi]-3\,\mathrm{Sin}[2\,(\phi0+\psi)])+\\&\frac{1}{12}\,Bz\,(12+3\,\mathrm{Cos}[2\,\phi0]-12\,\mathrm{Cos}[\psi]-3\,\mathrm{Cos}[2\,(\phi0+\psi)]-6\,\psi\,\mathrm{Sin}[\psi]-6\,\psi\,\mathrm{Sin}[2\,\phi0+\psi])\end{aligned}$$

Ws[ψ_] = Collect[Simplify[$\int_0^{\psi}$ Uz[x] Sin[φ0 + x] dx], {Az, Bz, Cz, Dz}]

$$\begin{aligned}&\frac{1}{12}\,Cz\,(-12\,\mathrm{Cos}[\phi0]+12\,\mathrm{Cos}[\phi0+\psi]-6\,\psi^2\,\mathrm{Cos}[\phi0+\psi]+12\,\psi\,\mathrm{Sin}[\phi0+\psi])+\\&\frac{1}{12}\,Dz\,(12\,\psi\,\mathrm{Cos}[\phi0+\psi]-2\,\psi^3\,\mathrm{Cos}[\phi0+\psi]+12\,\mathrm{Sin}[\phi0]-12\,\mathrm{Sin}[\phi0+\psi]+6\,\psi^2\,\mathrm{Sin}[\phi0+\psi])+\\&\frac{1}{12}\,Bz\,(6\,\psi+6\,\psi\,\mathrm{Cos}[\psi]+6\,\psi\,\mathrm{Cos}[2\,\phi0+\psi]+3\,\mathrm{Sin}[2\,\phi0]-12\,\mathrm{Sin}[\psi]-3\,\mathrm{Sin}[2\,(\phi0+\psi)])+\\&\frac{1}{12}\,Az\,(12-3\,\mathrm{Cos}[2\,\phi0]-12\,\mathrm{Cos}[\psi]+3\,\mathrm{Cos}[2\,(\phi0+\psi)]-6\,\psi\,\mathrm{Sin}[\psi]+6\,\psi\,\mathrm{Sin}[2\,\phi0+\psi])\end{aligned}$$

```
Vz[ψ_] = Collect[Simplify[∫_0^ψ ComplexExpand[Pz[x] Sin[ϕ0 + x]] dx], {Az, Bz, Cz, Dz}]
```

$$\frac{1}{4} Bz\,(-2 - Cos[2\,\phi0] + 2\,Cos[\psi] - Cos[2\,(\phi0 + \psi)] + 2\,Cos[2\,\phi0 + \psi]) + \frac{1}{4} Cz\,(-4\,\psi\,Cos[\phi0 + \psi] - 4\,Sin[\phi0] + 4\,Sin[\phi0 + \psi]) + \frac{1}{4} Dz\,(-4\,Cos[\phi0] + 4\,Cos[\phi0 + \psi] - 2\,\psi^2\,Cos[\phi0 + \psi] + 4\,\psi\,Sin[\phi0 + \psi]) + \frac{1}{4} Az\,(2\,\psi - Sin[2\,\phi0] - 2\,Sin[\psi] - Sin[2\,(\phi0 + \psi)] + 2\,Sin[2\,\phi0 + \psi])$$

```
Mz[ψ_] = Collect[Simplify[∫_0^ψ Vz[x] dx], {Az, Bz, Cz, Dz}]
```

$$\frac{1}{8} Az\,(-4 + 2\,\psi^2 + 3\,Cos[2\,\phi0] + 4\,Cos[\psi] + Cos[2\,(\phi0 + \psi)] - 4\,Cos[2\,\phi0 + \psi] - 2\,\psi\,Sin[2\,\phi0]) + \frac{1}{8} Cz\,(16\,Cos[\phi0] - 16\,Cos[\phi0 + \psi] - 8\,\psi\,Sin[\phi0] - 8\,\psi\,Sin[\phi0 + \psi]) + \frac{1}{8} Dz\,(-8\,\psi\,Cos[\phi0] - 16\,\psi\,Cos[\phi0 + \psi] - 24\,Sin[\phi0] + 24\,Sin[\phi0 + \psi] - 4\,\psi^2\,Sin[\phi0 + \psi]) + \frac{1}{8} Bz\,(-4\,\psi - 2\,\psi\,Cos[2\,\phi0] - 3\,Sin[2\,\phi0] + 4\,Sin[\psi] - Sin[2\,(\phi0 + \psi)] + 4\,Sin[2\,\phi0 + \psi])$$

```
Sz[ψ_] = Collect[Simplify[∫_0^ψ Uz[x] dx], {Az, Bz, Cz, Dz}]
```

$$\frac{Cz\,\psi^3}{6} + \frac{Dz\,\psi^4}{24} + Bz\left(Cos[\phi0] - \frac{1}{2}\,\psi^2\,Cos[\phi0] - Cos[\phi0 + \psi] - \psi\,Sin[\phi0]\right) + Az\left(\psi\,Cos[\phi0] + Sin[\phi0] - \frac{1}{2}\,\psi^2\,Sin[\phi0] - Sin[\phi0 + \psi]\right)$$

```
Tz[ψ_] = Collect[Simplify[∫_0^ψ Sz[x] dx], {Az, Bz, Cz, Dz}]
```

$$\frac{Cz\,\psi^4}{24} + \frac{Dz\,\psi^5}{120} + Az\left(\frac{1}{2}\,(-2 + \psi^2)\,Cos[\phi0] + Cos[\phi0 + \psi] + \psi\,Sin[\phi0] - \frac{1}{6}\,\psi^3\,Sin[\phi0]\right) + Bz\left(-\frac{1}{6}\,\psi\,(-6 + \psi^2)\,Cos[\phi0] + Sin[\phi0] - \frac{1}{2}\,\psi^2\,Sin[\phi0] - Sin[\phi0 + \psi]\right)$$

```
Gz[ψ_] = Collect[Simplify[∫_0^ψ x Pz[x] dx], {Az, Bz, Cz, Dz}]
```

$$\frac{Cz\,\psi^3}{3} + \frac{Dz\,\psi^4}{8} + Az\left(-\psi\,Cos[\phi0 + \psi] - Sin[\phi0] - \frac{1}{2}\,\psi^2\,Sin[\phi0] + Sin[\phi0 + \psi]\right) + Bz\left(\frac{1}{2}\,(-2 - \psi^2)\,Cos[\phi0] + Cos[\phi0 + \psi] + \psi\,Sin[\phi0 + \psi]\right)$$

```
Jz[ψ_] = Collect[Simplify[∫_0^ψ Gz[x] dx], {Az, Bz, Cz, Dz}]
```

$$\frac{Cz\,\psi^4}{12} + \frac{Dz\,\psi^5}{40} + Bz\left(-\frac{1}{6}\,\psi\,(6 + \psi^2)\,Cos[\phi0] - \psi\,Cos[\phi0 + \psi] - 2\,Sin[\phi0] + 2\,Sin[\phi0 + \psi]\right) + Az\left(2\,Cos[\phi0] - 2\,Cos[\phi0 + \psi] - \psi\,Sin[\phi0] - \frac{1}{6}\,\psi^3\,Sin[\phi0] - \psi\,Sin[\phi0 + \psi]\right)$$