# Combined method to extract spectroscopic information

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Spectroscopic factors (SF) play an important role in nuclear physics and astrophysics. The traditional method of extracting SF from direct transfer reactions suffers from serious ambiguities. We discuss a modified method which is based on including the asymptotic normalization coefficient (ANC) of the overlap functions into the transfer analysis. In the modified method the contribution of the external part of the reaction amplitude, typically dominant, is fixed and the SF is determined from fitting the internal part. We illustrate the modified method with $(d,p)$ reactions on $^{208}$Pb, $^{12}$C, and $^{84}$Se targets at different energies. The modified method allows one to extract the SF, which do not depend on the shape of the single-particle nucleon-target interaction, and has the potential of improving the reliability and accuracy of the structure information. This is specially important for nuclei on dripline, where not much is known.

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SF were introduced by the shell model formalism and are typically related to the shell occupancy of a state $n$ in one nucleus relative to a state $m$ in a nearby nucleus [1]. Today, phenomenological SF are extensively used in a variety of topics, from nuclear reactions to astrophysics or applied physics, yet the procedure for their extraction from the data has remained essentially the same for decades. For more than forty years since the dawn of nuclear physics, direct transfer reactions, such as $(d,p)$, $(d,t)$, $(^3\mathrm{He},d)$, $(^3\mathrm{He},\alpha)$, have been the central tool to determine SFs [2–4]. Extracting SFs with good precision from data is very important to test the validity of today's many body theories. For conventional nuclei there are many experiments available providing SFs, which are often lower than those predicted by shell model [1]. Electron-induced knockout or electron scattering is supposed to provide a better accuracy in extracting SFs than transfer [5, 6]. However, for exotic nuclei near or on the driplines, transfer reactions are a unique tool and, hence, can have a large impact in the programs of the new generation rare isotope laboratories. Given the experimental difficulties faced with measurements on the driplines, it is crucial to have a reliable method for analyzing and extracting useful information from each single data set.

Usually, transfer angular distributions are analyzed within the framework of the distorted-wave Born approximation (DWBA). The SF determined by normalizing the calculated DWBA differential cross section to the experimental one (e.g. [7–9]) is compared with the SF predicted by shell model. Even when error bars in the experimental cross section are low, the uncertainty of the extracted SF resulting from the normalization of the DWBA cross section is often large, regardless of whether it agrees with the shell model prediction. The reasons for this inaccuracy are typically: i) optical potentials ambiguity, ii) the inadequacy of the DWBA reaction theory, or iii) the dependence on the single-particle potential parameters. The first point has been object of a recent systematic study [10]. The second point needs to be addressed case by case, and examples of improved reaction models are the coupled channel Born approximation (e.g. [11]) or the continuum discretized coupled channel method (e.g. [12]). This work will critically review the standard procedure of extracting SFs from transfer reactions focusing on the third point; the modified method eliminates the dependence of the extracted SFs on the single-particle potentials, the main advantage of the method.

We will address a modified approach to spectroscopy from transfer reaction which includes the asymptotic normalization coefficient (ANC) in the analysis [2]. For simplicity, in the following formulation, we consider $A(d,p)B$ reaction and disregard spins (naturally these are included in the applications). The DWBA amplitude for this reaction is given by:

$$M = < \psi_f^{(-)} I_{An}^B | \Delta V | \varphi_{pn} \psi_i^{(+)} >, \qquad (1)$$

where $\Delta V = V_{pn} + V_{pA} - U_{pB}$ is the transition operator in the post-form, $V_{ij}$ is the interaction potential between $i$ and $j$, $U_{pB}$ is the optical potential in the final-state. The distorted waves in the initial and final states are $\psi_i^{(+)}$ and $\psi_f^{(-)}$, $\varphi_{pn}$ is the deuteron bound-state wave function and $I_{An}^B(\mathbf{r})$ is the overlap function of the bound-states of nuclei $B$ and $A$ which depends on $\mathbf{r}$, the radius-vector connecting the center of mass of $A$ with $n$. The overlap function is not an eigenfunction of an Hermitian Hamiltonian and is not normalized to unity [13]. The square norm of the overlap function gives a model-independent definition of the SF:

$$S = N < I_{An}^B | I_{An}^B > . \qquad (2)$$

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Here, $N$ is the antisymmetrization factor in the isospin formalism (N will be included in the overlap function from now on).

The leading asymptotic term of the radial overlap function (for $B = A + n$) is

$$I^B_{An(lj)}(r) \overset{r>R}{\approx} C_{lj}\, i\kappa\, h_l(i\kappa r), \qquad (3)$$

where $h_l(i\kappa r)$ is the spherical Bessel function, $\kappa = \sqrt{2\,\mu_{An}\,\varepsilon_{An}}$, $\varepsilon_{An}$ is the binding energy for $B \to A + n$, and $\mu_{An}$ is the reduced mass of $A$ and $n$. Similarly, the asymptotics of the neutron single-particle wave function is $\varphi_{An(n_r lj)}(r) \overset{r>R}{\approx} b_{n_r lj}\, i\kappa\, h_l(i\kappa r)$, where $n_r$ is the principle quantum number. The asymptotic behaviour is valid beyond $R$, the channel radius. It is clear that, in the asymptotic region, the overlap function is proportional to the single particle wave function. The normalization $C_{lj}$ introduced in Eq.(3) is the ANC which relates to the single-particle ANC (SPANC) $b_{n_r lj}$ by $C_{lj} = K_{n_r lj}\, b_{n_r lj}$, where $K_{n_r lj}$ is an asymptotic proportionality coefficient. It is standard practice to assume that the proportionality between the overlap function and the single particle function extends to all r values

$$I^B_{An(lj)}(r) = K_{n_r lj}\, \varphi_{An(n_r lj)}(r). \qquad (4)$$

Since $\varphi_{An(n_r lj)}(r)$ is normalized to unity, this approximation (Eq. 4) implies that $S_{lj} = K^2_{n_r lj}$. We have to emphasize, however, that the overlap function in the interior is nontrivial and may well differ from the single particle wavefunction. Approximating the radial dependence of the overlap function as described above leads to the DWBA amplitude $M = K_{n_r lj} < \psi^{(-)}_f \varphi_{An(n_r lj)} | \Delta V | \varphi_{pn}\, \psi^{(+)}_i >$. Normalizing the calculated DWBA cross section,

$$\sigma^{DW} = | < \varphi_{An(n_r lj)} | \Delta V | \varphi_{pn}\, \psi^{(+)}_i > |^2 \qquad (5)$$

to the experimental data provides the phenomenological SF $S_{lj} = K^2_{n_r lj}$. Assuming that Eq.(4) is valid for all r, we can infer from Eq.(2) that the main contribution to the norm of the overlap function comes from the nuclear interior.

In order to make the dependence on the SPANC more explicit, we split the reaction amplitude into an interior part and an exterior part:

$$M \;=\; K_{n_r lj}\, \tilde{M}_{int}[b] + K_{n_r lj}\, b_{n_r lj}\, \tilde{M}_{ext}, \qquad (6)$$

where the internal part of the matrix element $\tilde{M}_{int}[b_{n_r lj}] = < \psi^{(-)}_f \varphi_{An(n_r lj)} | \Delta V | \varphi_{pn}\, \psi^{(+)}_i >_{r<R}$ depends on $b_{n_r lj}$ through the bound state wavefunction $\varphi_{An(n_r lj)}$, while the external part $\tilde{M}_{ext} = < \psi^{(-)}_f\, i\kappa\, h_l(i\kappa r) | \Delta V | \varphi_{pn}\, \psi^{(+)}_i >_{r>R}$ does not depend on $b_{n_r lj}$. Here, $R$ is the channel radius taken so that for $r > R$ the overlap function can be approximated by its asymptotic form Eq.(3) (R is only used to illustrate the method as in the end this separation is not required). The contribution from the nuclear exterior is fixed by the ANC, whereas the SF determines the normalization of the internal part of the radial matrix element. Since transfer reactions are dominantly peripheral, SFs can only be extracted from transfer reactions due to a small contribution from the nuclear interior. We now introduce the ANC into the DWBA cross section:

$$\frac{\mathrm{d}\,\sigma^{DW}}{\mathrm{d}\Omega} = C^2_{lj}\, \frac{\sigma^{DW}}{b^2_{n_r lj}}. \qquad (7)$$

Introducing Eq.(6) into Eq.(7) and dividing by $C^2_{lj}$, we arrive at a function $R^{DW}(b)$

$$R^{DW}(b_{n_r lj}) = |\frac{\tilde{M}_{int}[b]}{b_{n_r lj}} + \tilde{M}_{ext}|^2. \qquad (8)$$

Note that the single-particle ANC $b_{n_r lj}$ itself is a function of the geometrical parameters of the bound state $n - A$ nuclear potential $(r_0, a)$ which are, a priori, not known. If the ANC and the cross section for the (d,p) reaction have been measured, the experimental counterpart of $R^{DW}$, $R^{exp} = \frac{\mathrm{d}\,\sigma^{exp}}{\mathrm{d}\Omega}/C^2_{lj}$ can be experimentally fixed. Then, imposing the equality

$$R^{exp} = R^{DW}(b_{n_r lj}), \qquad (9)$$

will provide the correct $b_{nlj}$ and consequently the SF $S_{lj} = C^2_{lj}/b^2_{nlj}$.

At this stage, a few points should be made clear. First of all, for specific optical potentials, Eq. (5) depends on two independent parameters, $S_{lj}$ and $b_{nlj}$. In the standard approach, to evaluate this cross section, the second parameter is fixed by arbitrarily choosing the bound state $n - A$ potential geometry. Thus, the extracted product $S_{lj}\, b^2_{n_r lj}$ does not coincide necessarily with the correct ANC. Since the ANC determines the normalization of the external part of the DWBA amplitude, in the standard approach the SF is determined by an unrealistic variation of the external contribution. In the modified method here discussed, since the contribution of the external part is fixed through the correct ANC, the whole DWBA procedure loses this artificial degree of freedom.

Secondly, if the reaction is peripheral, i. e. the first term in Eq. (6) is negligible, one can determine the ANC. So, the modified approach makes use of two experiments: the first to fix the ANC, the second to determine the SF consistent with that ANC. In present experiments and with the new generation of rare isotope facilities, ANCs can be determined with 5% accuracy. Since the determination of the SF comes from the internal region, the second experiment needs to be performed at a beam energy for which the contribution from the interior is significant. The higher the contribution of the internal region, the stronger the dependence on $b_{n_r lj}$ in $R^{DW}(b_{n_r lj})$ and the smaller the uncertainty of the extracted SF, although a balance needs to be found since large interior contributions may not be well describe by DWBA. The DWBA

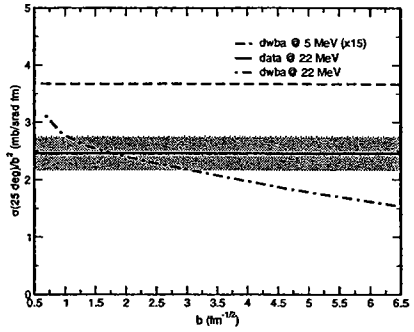


FIG. 1: Cross section for $^{208}$Pb(d,p)$^{209}$Pb(g.s.) at 22 MeV and the dependence on the single particle parameters: experimental value (solid line), experimental error bar (shaded area) and the DWBA prediction (dot-dashed).

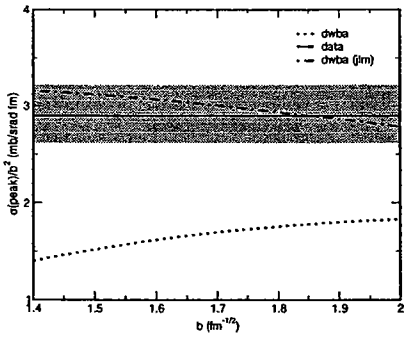


FIG. 2: Cross section for $^{12}$C(d,p)$^{13}$C(g.s.) at 51 MeV and the dependence on the single particle parameters: experimental value (solid line), experimental error bar (shaded area) and the DWBA prediction (dot-dashed).

differential cross section near the main peak of the angular distribution and, correspondingly, $R^{DW}(b_{n_rlj})$ are the functionals of the single-particle ANC $b_{n_rlj}$. One given $b_{n_rlj}$ can be produced by an infinite number of single-particle potentials, local and non-local. However, the dependence of $\mathrm{d}\sigma^{DW}/\mathrm{d}\Omega$ or $R^{DW}(b_{n_rlj})$ on the shape of the single-particle potential is minor. Hence, the extracted SF in the modified method does not depend on the single-particle potential. We illustrate the method presenting three different applications: i) $^{209}$Pb, ii) $^{13}$C and iii) $^{85}$Se. We will drop the subscripts on $b$ for simplicity.

Let us consider the reaction $^{208}$Pb(d,p)$^{209}$Pb from [12]. Although the ANC for $<^{209}\mathrm{Pb}|^{208}\mathrm{Pb}>$ is not published, it can be determined from the sub-Coulomb reaction [14] $^{208}$Pb($^{13}$C,$^{12}$C)$^{209}$Pb as the other vertex $<^{13}C|^{12}C>$ is well known [15]. Sub-Coulomb reactions are extremely peripheral and insensitive to details of the optical potentials. For this reason they present an excellent probe for extracting the ANC accurately. From [14] we obtain an ANC $C^2_{g9/2} = 2.15(0.16)\mathrm{fm}^{-1}$ for $^{209}$Pb. Then using $^{208}$Pb(d,p)$^{209}$Pb data at $E_d = 22$ MeV [12] we obtain $R^{exp} = 2.46(0.31)$ fm mb/srad, where the error bar is calculated based on both, the ANC and the cross sections errors, taken as independent. The experimental data in [12] has 1% accuracy but is taken only down to $\theta_{cm} = 35°$ whereas the peak of the DWBA distribution is at $\theta_{cm} = 25°$. We extrapolate the data based on the shape predicted by DWBA and include a 10% error in the cross section to account for this difference. Measurements at 25° could improve the error bar in $R^{exp}$ considerably. We next perform a series of finite range DWBA calculations for $^{208}$Pb(d,p)$^{209}$Pb ($E_d = 22$ MeV), using the optical potentials from [16]. The adiabatic prescription [17] was used to take into account deuteron breakup which is important for this reaction. The Reid-soft-core potential was used for the deuteron wavefunction, as well as in all other examples. For illustration purposes, we use a Woods Saxon well to generate the $^{208}Pb+n$ single-particle wavefunctions and obtain a range of SPANCs $b$ by varying the single particle parameters $(r_0, a)$ and adjusting the depth to reproduce the correct binding for the $2g_{9/2}$ in each case. We use the same s.o. strength as that in [14] although the s.o. strength does not affect the final result.

The results of our calculations $R^{DW}$ (dot-dashed line) and the experimental value $R^{exp}$ (solid line and shaded area) are presented, as a function of b, in Fig. 1. From $R^{exp}$ one finds $b = 1.82$ fm$^{-1/2}$ and S=0.74. It is worth noting that in the standard approach typical parameters $(r_0, a) = (1.2, 0.6)$ fm, produce $b = 1.34$ fm$^{-1/2}$. The direct comparison of the DWBA cross section using $(r_0, a) = (1.2, 0.6)$ fm, with the data, give S=0.866 and consequently, $C^2 = 1.56\,\mathrm{fm}^{-1}$, beyond the experimental range. As pointed out before, in the standard approach the SF is determined at the cost of an artificial ANC.

The beam energy of 22 MeV is above the Coulomb barrier, thus the reaction is not peripheral. This can be seen in Fig. 1 through the slope of the dot-dashed curve. In fact for this particular energy, the interior contribution is around 10%. The uncertainty in $b \in [1.1, 3.1]$ fm$^{-1/2}$ propagates into a large uncertainty in $S \in [0.3, 2.2]$. This is due to the fact that the contribution from the interior at this energy is still small. The scaling factor relating the uncertainty of S with that of b is $(|M_>|/|M_<|)^2$. The smaller the contribution from the interior, the smaller the accuracy with which the SF can be determined.

Also in Fig. 1 we show the results for $R^{DW}$ corresponding to the calculation at $E_d = 5$ MeV (dashed line). This is to illustrate that, at sub-Coulomb energies, the reaction becomes completely peripheral and the dependence on $b$ disappears. Measurements at these energies could provide $C^2_{g_{9/2}}$ with accuracy $< 5\%$. In addition, measurements at higher energy ($> 30$ MeV) would increase the slope of $R^{DW}(b)$ and decrease further the error on the extracted SF.

Another standard case is the $^{12}$C(d,p)$^{13}$C reaction, for which many data sets are conveniently compiled in a recent publication [10]. We studied three cases (8.9 MeV, 30 MeV and 51 MeV), using the same JLM optical potentials as [10]. We perform a series of finite range DWBA calculations varying the $1p_{1/2}$ $^{12}$C-n single particle parameters, in order to obtain $R^{DW}(b)$ as described before. Results for the less peripheral case (51

MeV) are plotted in Fig. 2 (dotted-dashed line). We take the data from [10] and the ANC from [15], to obtain $R^{exp} = \frac{\sigma(2.5^o)}{C^2_{1,1/2}} = 2.92(0.35)$ fm mb/srad. An $S = 0.66$ (shell model) would require $b = 1.89$ which is contained in our results. However, such a conclusion is misleading. Fig. 2 shows that even for this relatively large energy, the dependence of $R^{DW}$ on $b$ is weak. Consequently, it is not possible to extract a SF.

It was pointed out in [10] that the deuteron breakup is important for this reaction and should be taken into account. To emphasize this fact, we compare our results using the adiabatic deuteron potential [17] from [10] (dot-dashed line in Fig. 2) with those obtained using an optical potential fitted to the deuteron elastic scattering (dotted line in Fig. 2). The disagreement is very large. Interestingly, the method here described is also able to detect inadequate optical potential parameterizations.

Oak-Ridge has developed a program to measure a series of inverse kinematics (d,p) reactions for nuclei on the neutron dripline [18]. As one of the nuclei in the program is $^{85}$Se, we have performed exploratory calculations for $^{84}$Se(d,p)$^{85}$Se. We take global parameterizations for the optical potentials [16] and perform a series of calculations varying the single particle parameters. We compare the dependence of $R^{DW}$ on $b$ for a range of energies $E_d = 4 - 100$ MeV. We verify that, expectedly, the dependence on $b$ increases with beam energy. We find that Oak-Ridge energies (10 MeV/A) are adequate to determine ANCs but not SFs. However, a facility that allows for the production of $^{84}$Se at $E > 25$ MeV/A (such as NSCL-MSU, GANIL or RIKEN) could provide accurate spectroscopic information.

In conclusion, we have presented an alternative method to extract SFs, taking into account the sensitivity of the transfer data to the interior part of the overlap function and combining that information with the ANC. Transfer data can only become useful within this method if it has a significant contribution from the interior, and is well described through a one-step DWBA formalism. The balance between these two conditions is not a trivial one. By reducing the error bars in both the measured transfer cross section and the ANC, this prescription determines the single particle asymptotics and from it, a SF with reduced uncertainty. The ANC needs to be determined independently; it can be pinned down accurately with the same transfer reaction at sub-Coulomb energies or using heavy-ion induced reactions, both safely peripheral. Note that uncertainties due to optical potentials and higher order effects need to be assessed independently, as this work focuses on the single particle parameter uncertainties only.

The method here presented has the potential of reducing the uncertainty in the overlap function considerably. However it still assumes that the interior part has a Woods Saxon single particle wavefunction shape. This has been corroborated by recent Green's Function Monte Carlo calculations on light nuclei [19]. Even if there were non-localities of the single particle potential this would affect mostly the deep interior and thus would not be visible in the transfer reactions.

Results for (d,p) on $^{208}$Pb were used to illustrate the method. We discussed previous analyzes of (d,p) reactions on $^{12}$C, and showed the limitations. We have also demonstrated that this method can rule out inadequate choices of optical potentials. Considering specific future experiments, we have performed exploratory calculations for (d,p) on $^{84}$Se. This method will become useful for a broad variety of transfer experiments in the field of rare isotopes. The same method can equally be used for transfer to excited states. These same ideas can be extended to other reactions, in particular breakup reactions which also have an impact on Astrophysics. Finally, it would be helpful if the state-of-the-art reaction codes would incorporate the formalism discussed.

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