Smooth Approximation Description of Longitudinal Beam Dynamics in Superconducting Linacs

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DRAFT

Abstract

Smooth Approximation (SA) can be successfully applied to the charged particle beam dynamics in superconducting (SC) independently phased cavity (IPC) linac not only for the consistent description of the transverse optics [1,2], but also for the longitudinal beam dynamics as well. The new derivation of the SA formalism in the case of the small longitudinal oscillations in such a linac is detailed in the present paper. The comparison of the analytic prediction with the numerical simulation results received using LANA code [3] is shown. The possible applicability of the proposed approach is outlined.

1. SA derivation for the longitudinal oscillations.

As per [1] we can write for the longitudinal motion of the charged particle in the presence of the accelerating field the following system of the first order equations:

$$\begin{cases} \dfrac{d\psi}{dt} = \dfrac{\omega}{m_0 c^2 \cdot \beta^2 \cdot \gamma^3} \cdot p_\psi \\ \dfrac{dp_\psi}{dt} = Z/A \cdot e \cdot E_Z(t) \cdot \beta \cdot c \cdot \left[\cos(\varphi_0) - \cos(\psi + \varphi_0)\right] \end{cases} \qquad (1),$$

where: t is independent time variable; $p_\psi = W_0 - W$ – reversed particle's kinetic energy deviation from the reference; $\psi = \varphi - \varphi_0$ – particle's rf phase deviation from the reference; $\varphi = \omega \cdot t$ – rf phase of the particle with respect to the maximum of the accelerating field; $\omega = 2\pi \cdot f$, and f are cyclic and normal frequency of the rf field respectively; $\beta = v/c$ and v – relative and absolute velocity of the particle; c – speed of light; m_0 – rest mass of the particle; Z/A – particle's charge-to-mass ratio; e – charge of electron; and $E_Z(t)$ – accelerating rf field amplitude profile, that the particle experiences along it's trajectory.

The equations (1) neglects the Bessel function radial dependence of the accelerating field [1], but this approximation is still is very good for small deviations from the axis. To simplify this system of equations we can additionally assume:

1. ψ is small, so $\left[\cos(\varphi_0)-\cos(\psi+\varphi_0)\right]\approx\psi\cdot\sin(\varphi_0)$;
2. $\beta\approx const$ along the considered linac segment (the motion is adiabatic, i.e. the acceleration is sufficiently small).

Assuming that the accelerating field profile is periodic, and after changing to the dimensionless time variable $\tau=\frac{\beta\cdot c}{S}t$, where S is the longitudinal length of the considered period of the linac channel, and expressing $E_Z(t)=E_Z(\tau)=E_0T\cdot E(\tau)$, where $E(\tau)$ is the field profile normalized to 1, we can rewrite (1) as a single second order ordinary differential equation:

$$\frac{d^2\psi}{d\tau^2}+Q(\tau)\cdot\psi=0 \qquad (2),$$

where:

$$Q(\tau)=\eta\cdot E(\tau); \qquad (3),$$

and

$$\eta=-\frac{2\pi\cdot f\cdot S^2\cdot Z/A\cdot e\cdot E_0T\cdot\sin(\varphi_0)}{c\cdot m_0c^2\cdot(\beta\cdot\gamma)^3} \qquad (4).$$

The equation of this kind with the periodic coefficients: $Q(\tau+1)=Q(\tau)$, are called Mathieu equations [4]. One of the methods to solve such an equation is “smooth approximation” (SA) approach, which is actually a particular case of the first approximation of the “averaging” method, developed by N.N.Bogoliubov and Y.A.Mitropolsky [5]. This method can be used for the description of the longitudinal periodic motion of the charged particles in the accelerating systems with phase alternation [6]. In the present paper author proposes to use this approach for the SC IPC linac periodic accelerating channel longitudinal beam dynamic characteristics description.

The SA method is valid if the condition:

$$S/L_{foc} << 1 \qquad (5)$$

is fulfilled [1], where the L_{foc} is the characteristic “focusing length” of the “lens” – the bunching distance of the monochromatic beam after the accelerating field region (cavity) with the given parameters in this model. In the other words – this condition means that the frequency of the external force from the lattice is much higher then the eigenfrequency of the oscillating system.

We can seek the solution of (2) in form of:

$$\psi(\tau)=\left[1+q(\tau)\right]\cdot X(\tau) \qquad (6),$$

where: the $q(\tau)$ is the fast component of the solution, and $X(\tau)$ – is the slow one. Defining the $q(\tau)$ function with:

$$\frac{d^2q}{d\tau^2} = \langle Q \rangle - Q(\tau) \qquad (7),$$

where:

$$\langle Q \rangle = \int_{\tau}^{\tau+1} Q(\tau) d\tau \qquad (8),$$

and using the normalizations:

$$\left\langle \frac{dq}{d\tau} \right\rangle = \int_{\tau}^{\tau+1} \frac{dq}{d\tau}(\tau) d\tau = 0 \qquad (9),$$

and

$$\langle q \rangle = \int_{\tau}^{\tau+1} q(\tau) d\tau = 0 \qquad (10),$$

for defining of the arbitrary integration constants, we can find the explicit form of the function $q(\tau)$ after integrating twice the equation (7), for known form of $Q(\tau)$ from (3).

Substituting (6) in (2) and taking into account (9) and (10), and supposing that the slow component $X(\tau)$ is “smooth”, so that we can neglect it’s change on the considered period, we can arrive at:

$$\frac{d^2X}{d\tau^2} + \mu^2 \cdot X = 0 \qquad (11),$$

where we used the definition:

$$\mu^2 = \langle Q \rangle + \int_{\tau}^{\tau+1} q(\tau) \cdot Q(\tau) \cdot d\tau \qquad (12).$$

The eq. (11) is the equation of the harmonic oscillator and the value μ has a meaning of the cyclic frequency of the slow component of the solution: $X(\tau)$.

It is known [1,7] that the value of the frequency μ can be determined more accurately if instead of directly using (12) one would use the power series decomposition of the $\cos\mu$ for the small μ to determine the $\cos\mu$ from (12) as:

$$\cos\mu = 1 - \frac{1}{2}\mu^2 \qquad (13).$$

The given derivation is generally valid for arbitrary form of the $E(\tau)$ function.

The condition (5) can be expressed as:

$$\mu \ll 2\pi \qquad (14).$$

For practical purposes the accuracy of the SA is reasonable when [1]:

$$0.3 < \cos\mu < 1 \qquad (15),$$

or:

$$0 < \mu < 1.3 \qquad (16).$$

Let us apply the results derived above to the model case of the form of function $E(\tau)$.

1.1. Single sharp-edge localized accelerating field region model.

Let us assume that the $E(\tau)$ function can by represented in a sharp edge approximation as a square wave on the accelerating channel period as shown on the Fig. 1, where the following definition is used for the abscissa variable:

$$\xi = {L_{cavity}}\big/{S} \qquad (17).$$

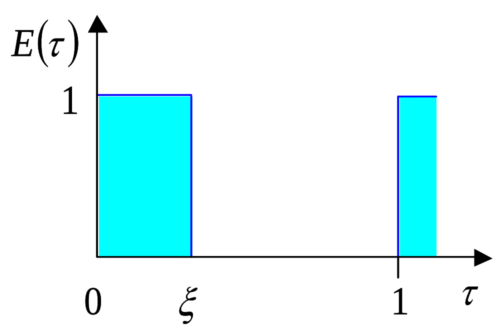

Fig. 1. Schematic representation of the model accelerating field profile on the period of the accelerating channel.

Knowing $Q(\tau)$ from (3) we can integrate (7) twice and for function $q(\tau)$ at:

$$q(\tau) = C_2 + C_1 \cdot \tau + \frac{1}{2} \cdot \eta \cdot \xi \cdot \tau^2 - \frac{1}{2} \cdot \eta \cdot \tau^2 \cdot \vartheta(\tau) \cdot \vartheta(\xi - \tau) \qquad (18),$$

where $\vartheta(\tau)$ is the Heavyside's step-function.

The arbitrary constants C_1 and C_2 can be determined from the conditions (9) and (10) as:

$$
\begin{aligned}
C_1 &= \frac{1}{2}\cdot\eta\cdot\xi^2 - \frac{1}{2}\cdot\eta\cdot\xi \\
C_2 &= \frac{1}{6}\cdot\eta\cdot\xi^3 - \frac{1}{4}\cdot\eta\cdot\xi^2 + \frac{1}{12}\cdot\eta\cdot\xi
\end{aligned}
\qquad (19).
$$

As the final expression for the $\cos\mu$ from (12) and (13) we can write:

$$
\cos\mu = 1 - \frac{1}{2}\cdot\eta\cdot\xi - \frac{1}{3}\cdot\eta^2\cdot\left(\frac{1}{8}\cdot\xi^2 - \xi^3 + \frac{7}{8}\cdot\xi^4\right) \qquad (20).
$$

2. Comparison of the SA analytic predictions and the LANA simulation results.

For comparison of the analytic prediction and the corresponding numerical model results several sets of LANA input files for simulation different versions of the proposed RIA Driver Linac were used [8]. Without going into the details of the different cases author wants just to present this comparison results, which shall be extended in the future studies.

Table 1 lists several compared cases of the corresponding longitudinal oscillation phase advances for the most critical front-end period of the SC RIA Driver Linac accelerating channel.

Table 1. The comparison of the longitudinal oscillations phase advance μ [deg] on the first period of the accelerating channel of the RIA Driver Linac at NSCL/MSU between LANA simulation results and the SA analytic predictions for several simulated cases [8].

	Case1	Case2	Case3
LANA	99°	84°	97°
SA	101°	88°	101°
Error	2 %	5 %	4 %

The expected accuracy of 5-8 % is demonstrated in all cases with slightly above the upper limit of the SA applicability margins which according to (16) is equal to:

$$
0^o < \mu < 75^o \qquad (21).
$$

One of the other simulated cases, that has $\mu \approx 141^o$ according to the LANA numerical model, is not adequately described by the proposed analytic model because of the extreme violation of the applicability region margins.

3. Applicability of the proposed formalism for the SC IPC linac design.

The derived analytic approach can be very useful at the design stage of the SC IPC linacs beam dynamics, when the numerical model for the appropriate accelerating structures and the parameters of the accelerating channel have to be chosen. The recipes from this approach can be a good starting point for the numerical model setup and for optimization of the linac channel parameters in the computer simulations. In particular such information can be found useful for special matching cells settings as described in [8].

The other important application of this approach can be the understanding of the limitations of different parameters of the accelerating channel of IPC in the SC linacs, such as geometric dimensions of the interior drifts in the cryostat and the allowable inter-cryostat drifts, as well as the adequate injection energy for the desired longitudinal frequency in the front-end of the SC linac and maximum limits for the effective accelerating voltages in the rf cavities, etc.

The beam quality in linacs can be deteriorated by the parametric resonances. Sufficiently good estimation of possible limitations on the transverse focusing optics parameters can be also predicted based on the proposed formalism.

The proposed approach simplifies the investigation and detailed analysis of the beam dynamics properties in the SC IPC linacs. The derived formalism can also be used in the beam dynamics computer simulation codes for developing of powerful fitting capabilities aimed at the optimization of the longitudinal beam dynamics in SC IPC linacs.

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