MICHIGAN STATE UNIVERSITY

CYCLOTRON LABORATORY

DETONATION ACROSS A TIMELIKE SURFACE FOR RELATIVISTIC SYSTEMS

L. P. CSERNAI

DECEMBER 1986

MSUCL-583

MSUCL-583
December 1986

Detonation Across a Timelike Surface for Relativistic Systems

L.P. Csernai

National Superconducting Cyclotron Laboratory†
Michigan State University
E Lansing, Michigan, USA
and

Central Research Institute for Physics*
Budapest, Hungary

Submitted to Sov. JETP

Abstract:

Phase transitions in high energy heavy ion collisions, in the early Universe and in other relativistic systems, may occur on a 3 dimensional hypersurface the points of which are not causally connected. It is shown that Taub's original derivation of the relativistic shock (or detonation) adiabat can be generalized in discontinuities across surfaces having a timelike normal vector, and it can be cast in a form universal for both spacelike and timelike discontinuities. An example is outlined of an implosion-explosion where the front switches from spacelike to timelike due to radiative heat transfer.

† Present address
* Permanent address

This work was supported by the U.S. Department of Energy under Contracts DE-AC03-76S00098, DE-AC02-79ER10364 and by the National Science Foundation under grant PHY-83-12245.

The theory of shock waves in relativistic fluid dynamics[1] has proven its importance through a number of elegant applications in cosmology[2-5] and, more recently, in relativistic heavy ion reactions.[6-10] Usually a shock wave is formed when matter is compressed very rapidly by an external force. Shock-like discontinuities would also be formed spontaneously when a system undergoes a first order phase transition during the expansion stage.[11] In all these cases, shock waves propagate with a velocity smaller than the light velocity. Therefore the world sheet swept out by the surface of the shock front draws a space-like hypersurface in the Minkowski space (Fig. 1a) (i.e. it has a space-like normal vector). The conservation laws across this surface lead to the Rankine-Hugoniot-Taub equation,[1] which relates the fluid variables (e.g. pressure, fluid velocity, conserved charge density) on the two sides of the shock front.

Recently it was pointed out[12] that under certain conditions the system may undergo a rapid bulk phase transition through a timelike surface of discontinuity. This happens when the system is diluted very rapidly and uniformly and the nucleation of bubbles occurs at many spatially adjacent points which are not causally connected to each other. An example of this is the inflationary universe scenario. In this case, the spacelike phase boundary, if it is smoothed out, becomes a timelike surface Σ (see Fig. 1b). The thickness τ of this transition zone would be determined by the bubble formation rate and the velocity of bubble growth. If τ is small enough compared with the characteristic time scale of the processes of interest, one may consider the phase transition as taking place through a structureless timelike surface. The purpose of this note is to give a general derivation of the Rankine-Hugoniot-Taub equation which is applicable both for the spacelike surface and the timelike surface of shock discontinuity. A simple example will illustrate the general utility of the result.

Let us denote the normal vector of the surface Σ by Λ^{μ}. Then from the normalization:

$$\Lambda_{\mu}\Lambda^{\mu} = \begin{cases} +1 - \text{timelike surface } \Sigma \\ -1 - \text{spacelike surface } \Sigma . \end{cases} \tag{1}$$

The physical state of the system can be characterized by the energy momentumtensor:

$$T^{\mu\nu} = w\, u^{\mu} u^{\nu} - p\, g^{\mu\nu} , \tag{2}$$

where the enthalpy density, $w = e+p$, is the sum of the energy density, e, and pressure, p, $u^{\mu} = (\gamma, \gamma \bar{v})$ is the four fluid velocity,normalized to $u^{\mu}u_{\mu}=1$ and $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the metric tensor. If we have a discontinuity across the surface we can label any given fluid dynamical quantity, say Q, on one side of the surface by the index "1"(Q_1) and on the other side by "2"(i.e.Q_2). Then the change of this quantity will be represented by

$$[Q] = Q_2 - Q_1 \quad . \tag{3}$$

Using this notation the conservation laws across the surface of the discontinuity take the form:

$$[R^{\mu}] = [\, T_{\mu\nu}\Lambda_{\nu}] = 0 \quad , \tag{4}$$

(energy and momentum conservation)

$$[j] = [\, n^{\mu} \Lambda_{\mu}] = 0 \quad , \tag{5}$$

(conservation of particle number). Systems with no conserved particles are discussed in Appendix I. Eq. (5) can be written for any independently conserved charge with the four current of the conserved charge. We can introduce an invariant scalar proper density $n = n^{\mu} u_{\mu}$ assuming that the flow is coupled to the conserved charge, and a quantity[2,3,6] $x = w/n^2$, which plays the same role as the specific volume V in the nonrelativistic theory. In fact, $x \to mV$ in the nonrelativistic limit (m is the particle mass).

In order to derive the equation of the detonation adiabat (which depends on thermodynamic quantities only) we have to eliminate the four velocity from eqs.(4) and (5). Since eq.(4) is a vector equation we can cast it into two independent equations by taking its projection into the normal direction of the surface Σ

$$[R^{\mu}]\, \Lambda_{\mu} = 0 \quad , \tag{6}$$

and projecting it into the surface by the projector $P^{\mu\nu} = g^{\mu\nu} - \Lambda^{\mu}\Lambda^{\nu}/(\Lambda^{\alpha}\Lambda_{\alpha})$:

$$[\, G^{\mu}] = [\, P^{\mu\nu} R_{\nu}] = 0 \quad . \tag{7}$$

Eq. (7) can be split into a scalar equation showing that the length of the vector G^{μ} is conserved

$$[\, G^{\mu} . G_{\mu}] = 0 \, , \tag{8}$$

and into an equation which requires the direction of the projection G^{μ} be unchanged, $[G^{\mu}/|G^{\mu}|] = 0$. After straightforward calculation from eqs. (5) and (6) we find that the current j across the surface satisfies

$$j^2 = [p]\,(\Lambda^\mu \Lambda_\mu)\,/\,[x] \quad . \tag{9}$$

Also, from eqs. (5) and (8), we obtain

$$j^2 = [wx]\,/\,[x^2](\Lambda_\mu \Lambda^\mu) \quad . \tag{10}$$

From the above two relations we can get immediately the equation of the Taub adiabat

$$[p]\,(x_1+x_2) = [wx] \quad , \tag{11}$$

in its already well known form. For given p_1 and x_1, this equation determines the relation between p_2 and x_2, if the equation of state is known. We note here that the normal vector of the surface Λ^μ entered in this equation only through $(\Lambda_\mu \Lambda^\mu)^2$, which is equal to 1 and hence dropped out! Thus the equation of the Taub adiabat is the same for spacelike and timelike surfaces of discontinuity. Yet, there exists an essential difference between timelike and spacelike surfaces of discontinuities.

To show this the shock and detonation adiabats are plotted in the (p,x) plane (Fig. 2). Here initial state is marked by "1" and the final state by "2". In the case of a normal shock, the adiabat determined by equation (11) goes through point "1" since the final state is described by the same equation of state as that of the initial state. If the equation of state changes in the final state due to a chemical reaction or a phase transition, the point "2" sits on the curve which does not pass through the original point "1". This curve is commonly called the detonation adiabat. We plotted here the detonation adiabat for an exotherm process. If it is endotherm process the curve lies below the point "1". The sector OA on this curve corresponds to detonation and the sectr A'O' to slow combustion (Deflagration). The hatched areas on this plane, where $[p]/[x]<0$, have previously been considered unphysical in the literature[9,3] based on nonrelativistic analogies[13], since the current j becomes imaginary in this region by eq.(9) for spacelike surfaces of discontinuity $(\Lambda_\mu \Lambda^\mu) = -1$. This region, however, can be reached with real j if one consideres a <u>timelike surface of the detonation front</u>. We note that timelike normal shock fronts cannot exist since the shock adiabat can never enter the timelike regions (hatched areas in Fig. 2). This can be easily seen, if we recall that the timelike surface can be locally Lorentz transformed into frame where its normal vector is $\Lambda^\mu = (1,0,0,0)$. In this system there is a sudden change in the pressure, p and x, but the density $N = n\gamma$ measured in this frame, remains unchanged (because $[j] = [n\gamma] = [N] = 0$). This is possible only if a spontaneous phase transition or chemical reaction takes place in the system, i.e., a detonation occurs.

Spontaneous detonation can occur, however, only if it satisfies the entropy increase law. Since the Poisson adiabat is parallel to the shock adiabat in point "1"[2], the higher entropy states are above point "1" and therefore only the upper quarter of the timelike detonation region can be reached in a physical process. In other words only exotherm detonations can take place spontaneously across a timelike surface. If there is a certain threshold (pressure, temperature) for the exotherm process to occur in the front, this might terminate the physically realizable timelike adiabat at some point inside AA'.

In a simple schematic model (see also Appendix II) one can demonstrate how a radiation dominated spacelike detonation can develop continuously into a timelike detonation in an implosion. Surround a spherical core of unit radius ($R=1$, $c=1$) by a fast burning outer shell. Ignite this shell on all sides at $t=0$. Part of the energy released radiates inwards and heats the core. The $T(r,t)$ = const. contour lines can be calculated easily if one neglects the opacity and compression of the core:

$$T(r,t) \propto \left\{ 0;\ \frac{t}{r}\left(\ln\frac{t}{1-r} - 1\right) + \frac{1-r}{r};\ \frac{t}{r}\ln\frac{1+r}{1-r} - 2; \right\}$$

$$\text{if } \{ t < 1-r;\ 1-r < t < 1+r;\ 1+r > t \};\ \text{respectively}$$

Let the core undergo an exotherm transition if the temperature reaches a critical value T_c. This transition then takes place on the surface $T(r,t) = T_c$. If the heating of the core to T_c is fast ($t \approx 2.5$) an essential part of the surface ($r \lesssim 0.5$, $2.3 \lesssim t \lesssim 2.5$) is a timelike detonation (Fig. 3). If the heating is slower the timelike part of the detonation shrinks to a smaller central region.

Summarizing these results it can be emphasized that using a general derivation for a discontinuity across an arbitrary surface in relativistic fluid flow we succeeded in extending the Rankine-Hugoniot-Taub equations into a new region not discussed before. This region was even considered being unphysical based on nonrelativistic analogies. The introduction of the concept of timelike detonations completes the relativistic theory of rapid combustion and condensation phenomena. This formalism should allow for a more unified conceptual understanding and for more transparent mathematical manipulations in problems of relativistic fluid dynamics as illustrated by the physical example in Fig. 3.

The author thanks Tatsuo Matsui and Igor Mishustin for their valuable contribution to this work. Enlightening discussions with George Chapline, Miklos Gyulassy and Joseph Kapusta are greatly appreciated.

Appendix I

Detonation and Deflagration fronts for QCD plasma and hadronic matter, with zero baryon charge.

Notation:

$u^\mu = (\gamma, \gamma\vec{v})$, $u_\mu = (\gamma, -\gamma\vec{v})$, $u^\mu u_\mu = +1$, $g^{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.
The normal vector of the discontinuity is Λ^μ, normalized as $\Lambda^\mu \Lambda_\mu = \pm 1$ (timelike, spacelike). In the local rest frame $\Lambda^\mu = (1,0,0,0)$ timelike, $\Lambda^\mu = (0,\hat{e})$ spacelike.

1. Parallel projection to Λ^μ:

$$[T^{\mu\nu}\Lambda_\mu \Lambda_\nu] = [w(u^\mu \Lambda_\mu)^2 - p\,\Lambda_\mu \Lambda^\mu] = 0. \qquad \text{(A1)}$$

2. Orthogonal projection to Λ^μ:

Introduce $G_\tau = T^{\mu\nu}\Lambda_\nu \Delta_{\tau\mu}$ where $\Delta^{\tau\mu} = g^{\tau\mu} - \dfrac{\Lambda^\tau \Lambda^\mu}{(\Lambda^\sigma \Lambda_\sigma)}$.
(G_τ is orthogonal to Λ_τ)

$$G^\mu = (w u_\tau (u_\nu \Lambda^\nu) - p\Lambda_\tau)\left(g^{\tau\mu} - \frac{\Lambda^\tau \Lambda^\mu}{(\Lambda^\sigma \Lambda_\sigma)}\right) = w\,(u^\tau \Lambda_\tau)u^\mu - w\,(u_\nu \Lambda^\nu)^2 \frac{\Lambda^\mu}{(\Lambda^\sigma \Lambda_\sigma)}.$$

Since $[G^\mu] = 0$ it follows that $[G^\mu G_\mu] = 0$ too, which leads to:

$$[w^2(u_\nu \Lambda^\nu)^2 - w^2 (u_\nu \Lambda^\nu)^4/(\Lambda^\sigma \Lambda_\sigma)] = 0. \qquad \text{(A2)}$$

Let us denote $w(u_\nu \Lambda^\nu)^2$ by $Q = w(u_\nu \Lambda^\nu)^2$ and the norm of the normal vector of the surface $\Lambda^\nu \Lambda_\nu$ by $N = (\Lambda^\nu \Lambda_\nu)$, then (A1), (A2) reads as

$$[Q] = N[p]\,,$$
$$[Q^2] = N[wp]\,.$$

Eliminating N one gets that

$$[p]\,(Q_2 + Q_1) = [wQ]\,, \qquad \text{(A3)}$$

and

$$Q_1 = -N\,(p_2 - p_1)(e_2 + p_1)\,/\,(e_2 - p_2 - e_1 + p_1)\,.$$

Consequently

$$(u^\mu \Lambda_\mu)_1^2 = \frac{N\,(p_2 - p_1)\,(e_2 + p_1)}{(e_2 - p_2 - e_1 + p_1)\,(e_1 + p_1)}\,, \qquad \text{(A4)}$$

on the other hand

$$(u^\mu \Lambda_\mu)_1^2 = \begin{cases} \gamma_1^2 \, v_1^2 \cos^2\theta_1 & : \text{spacelike} \\ \gamma_1^2 & : \text{timelike} \end{cases} \tag{A5}$$

In ref. 9 spacelike fronts were considered with $\theta_1 = 0$ only. From (A4,5) the velocity of the incoming matter (A1) in the front's frame v_1 is (if $\theta_1 = \theta_2 = 0$):

$$v_1^2 = (p_1 - p_2)(e_2 + p_1) / \{(e_1 - e_2)(e_1 + p_2)\} . \tag{A6}$$

The velocity v_1' in the timelike detonation is:

$$(v_1')^2 = (v_1)^{-2}$$

The relative speed of the incoming and outgoing matter both in spacelike and in timelike fronts are:

$$v_{12}^2 = v_{12}'^2 = (p_1 - p_2)(e_1 - e_2) / \{(e_1 + p_2)(e_2 + p_1)\} . \tag{A7}$$

From the condition that the entropy should increase with the entropy flux: $S^\mu = s\, u^\mu$

$$[S^\mu \Lambda_\mu] \geq 0 \ \rightsquigarrow \ \frac{s_2}{s_1} \geq \frac{T_2}{T_1} \, \frac{e_2 + p_1}{e_1 + p_2} \tag{A8}$$

for both types of detonations, for $1 \to 2$ process. Inserting the same equation of state into eqs. (A6-8) and performing the same analysis what is done in ref. 9 we obtain a domain where timelike shocks are realizable (Fig. 4).

Appendix II

Radiation dominated implosion

Consider a spherical peace of matter (E) which is sufficiently transparent for radiation. This matter undergoes an exotherm transition if its temperature exceeds T_c.

Surround this by an outer shell of fast burning explosive (H) which emits the radiation necessary to heat up E. Neglect the expansion of the outer shell inwards as well as the expansion of the core, so that the core radius is R = constant. Let us ignite the shell with synchronized clocks at $t_0 = 0$. Measure the length in units of R and t in units of R/c.

The shell emits Q heat per unit surface and unit time. Consequently if we neglect the opacity of the core, in a given point at r measured from the center of the sphere (assuming a small constant fraction C of the heat absorbed):

$$\frac{dQ}{dt} = C Q \int_0^t d\tau \, 2\pi \int_1^0 d\cos\theta \, (1+r^2-2r\cos\theta)^{-1} \, \delta\left(\tau - \sqrt{1+r^2-2r\cos\theta}\right) =$$

$$= C\, 2\pi Q\, r^{-1} \int_{1-r}^{a} \frac{d\tau}{\tau} = c\, 2\pi\, Q\, r^{-1} \left[\ln \tau\right]_{1-r}^{a} ,$$

where $a = \{ 1-r$ if $t \leq 1-r$; t if $1-r<t<1+r$; $1+r$ if $t>1+r \}$. (A9)

So the heat deposited per unit time is:

$$\frac{dQ}{dt} = \frac{2\pi C Q}{r} \begin{cases} \ln\{(1+r)/(1-r)\} ; & t > 1+r \\ \ln\{ t/(1-r)\} ; & 1-r<t<1+r \\ 0 ; & 1-r > t \end{cases} \quad \text{(A10)}$$

Neglecting compression and assuming a constant specific heat C_v, $dT \cong \frac{1}{C_v} dQ$ so

$$T(r,t) \cong \frac{1}{C_v} \int_0^t dt \, \frac{dQ}{dt} =$$

$$= \frac{2\pi c Q}{c_v r} \left\{ \begin{array}{ll} t\left\{ \ln \frac{1+r}{1-r} \right\} - 2r & ; t > 1+r \\ t\left\{ \ln \frac{t}{1-r} - 1 \right\} - 1 - r & ; 1-r < t < 1+r \\ 0 & ; t < 1-r \end{array} \right. . \quad \text{(A11)}$$

(so if $t>1+r$ then $T(r=0,t) \propto (t-1)$.) The surface of the discontinuity is characterized by the $T(r,t)T_c$ contour line. The tangent of this line is if $t>1+r$:

$$\left(\frac{\partial r}{\partial t}\right)_{T_c} = \left(\frac{\partial T}{\partial t}\right)_{T_c} \Big/ \left(\frac{\partial T}{\partial r}\right)_{T_c} =$$

$$= \ln \frac{1+r}{1-r} \Big/ \left\{ t \left[\frac{2}{1-r} - \frac{1}{r} \ln \frac{1+r}{1-r} \right] \right\} , \quad \text{(A12)}$$

so the point (t_c, r_c) where the spacelike and timelike parts of the surface meet: $\quad (\partial r/\partial t)_{T_c} = 1$,

$$t_c = \left\{ \left[(1-r_c) \ln \sqrt{\frac{1+r_c}{1-r_c}} \right]^{-1} - \frac{1}{r_c} \right\}^{-1} . \quad \text{(A13)}$$

For example: if $r_c = 0.5$ $t_c = 2.34$ and $T_c = 3.142$.
The center heat up time to T_c is $t = 2.57$.
This line $t=t_c(r)$ separates the Spacelike and Timelike branch of the discontinuity of $T(r,t) = T_c$.
The discontinuity initiates at $r=R$ and $t=0$ and it propagates first slowly inwards. Due to the radiative heat transfer it accelerates and at $r_c = t_c^{-1}(t_c(r))$ it developes smoothly into a timelike discontinuity (see Fig. 3).

If $T_c \cong \frac{4\pi c Q}{c_v}$ then this happens at about $r_c \approx 0.5\text{-}0.6$.

The same type of gradual development from spacelike into timelike detonation may occur in the last phase of ultra-relativistic heavy ion collisions. If we include radiative heat transfer in a scenario described in ref. 12 the transition from spacelike to timelike deflagration will be gradual. This, however, requires more involved numerical calculations.

Figure captions

Fig. 1 Comparison of spacelike (a) and timelike (b) surfaces of discontinuities characterized by their normal vector Λ^{μ} The timelike discontinuity might be formed from spontaneously formed expanding bubbles which after a time $\mathcal{T}$ coalesce (full line). If we smooth out this complicated surface a timelike surface will be obtained (dashed line). Note that in viscous flow the spacelike discontinuity has a finite width too, thus the substructure of the timelike front (b) may develop only if the bubble separation is large compared to the width of the spacelike fronts.

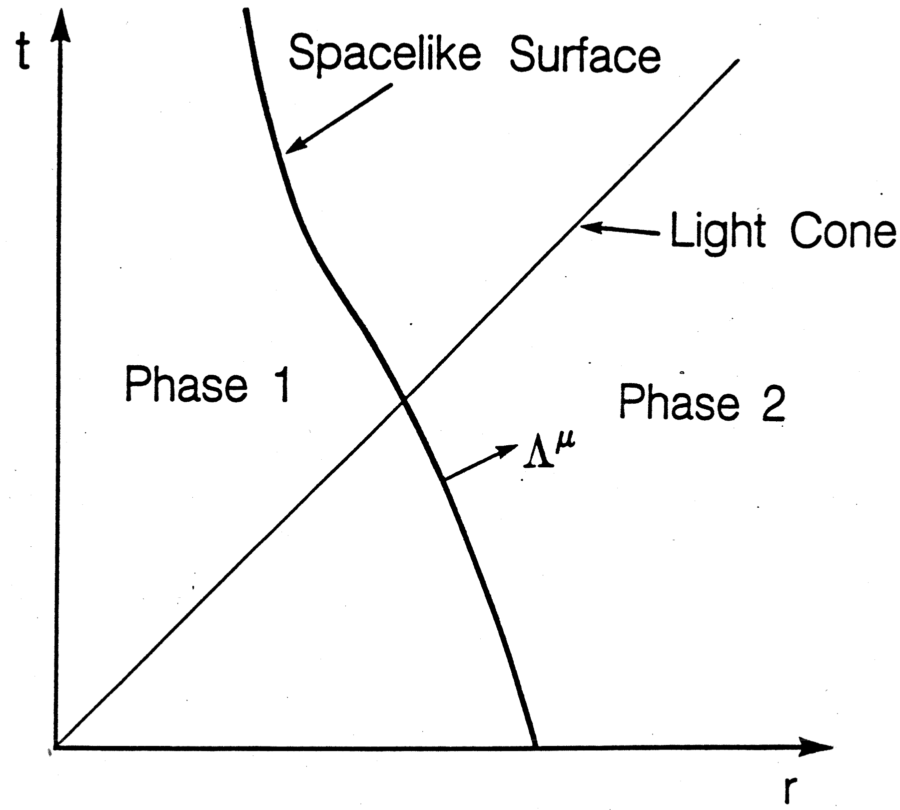
Fig. 2 Shock (a) and detonation (b) adiabats on the pressure p, generalized specific volume $x = w/n^2$ plane from an initial state "1". The final state "2" should lie on the adiabat. The tangent of the straight line connecting the initial and final points depends on the current across the front as $j^2 = \pm [p]/[x]$ for (timelike/spacelike) discontinuity. The section AA' of the detonation adiabat represents timelike discontinuities. O and O' are the Chapman-Jouguet points on the detonation adiabat[2]. A rapid combustion or condensation, OA, when accelerated due to the boundary conditions may go over smoothly into a timelike detonation through point A.

Fig. 3 The acceleration of the discontinuity due to radiation may lead to a smooth transition from a spacelike to timelike front across point A where the front propagates with the speed of light. This is possible since the fluid does not move with the front. The world lines of the fluid stay timelike (light lines), i.e. their velocity is $v < c$.

Fig. 4 Kinematic domains in which the continuity equations can be satisfied with physical flow velocities for a transition between quark matter of energy density e_2 and hadronic matter of energy density e_1. B is the bag constant. The unphysical domain is smaller than was obtained in ref. 9 (Fig. 4) because of the possibility of timelike detonations.

References

1. A.H. Taub, Phys. Rev. 74, 328 (1948)
2. K.S. Thorne, Astrophys. J. 179, 897 (1973)
3. P.J. Steinhardt, Phys. Rev. D25, 2074 (1982)
4. M.H. Johnson and C.F. McKee, Phys. Rev. D3, 858 (1971)
5. G. Lasher, Phys. Rev. Lett. 42, 1646 (1979);
   A.H. Guth, E.J. Weinberg, pPhys. Rev. D23, 876 (1981)
6. E.V. Shuryak, Phys. Reports 61, 71 (1980)
7. C. Van Hove, Z. Phys. C21, 93 (1983)
8. B.L. Friman, G. Baym and J.-P. Blaizot, Phys. Lett. 132B, 291 (1983)
9. M. Gyulassy, K. Kajantie, H. Kurki-Suonio and L. McLerran, Nucl. Phys. B237, 477 (1984)
10. L.P. Csernai, Phys. Rev. D29 1945 (1984)
11. This may occur as a condensation discontinuity in a rarefaction wave[8] or as a deflagration or detonation front of expanding bubbles formed in a supercooled metastable medium[3,9]. Here we call any surface of discontinuity "shock front". The distinction between these discontinuities and the normal shocks will be discussed later.
12. N.K. Glendenning and T. Matsui, Phys. Lett. 141B, 419 (1984)
13. L.D. Landau and E.M. Lifshitz, Fluid-Mechanics (Pergamon, 1959)


t
Spacelike Surface
Light Cone
Phase 1
Phase 2
Λ^μ
r

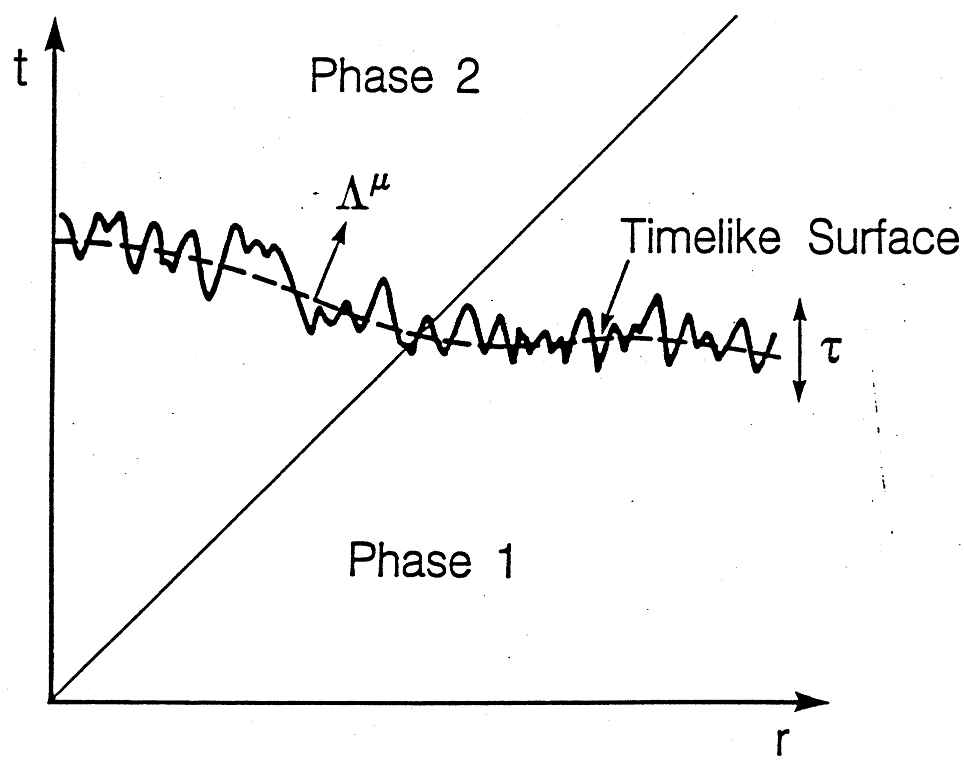

t
Phase 2
Λ^μ
Timelike Surface
τ
Phase 1
r

Fig. 1

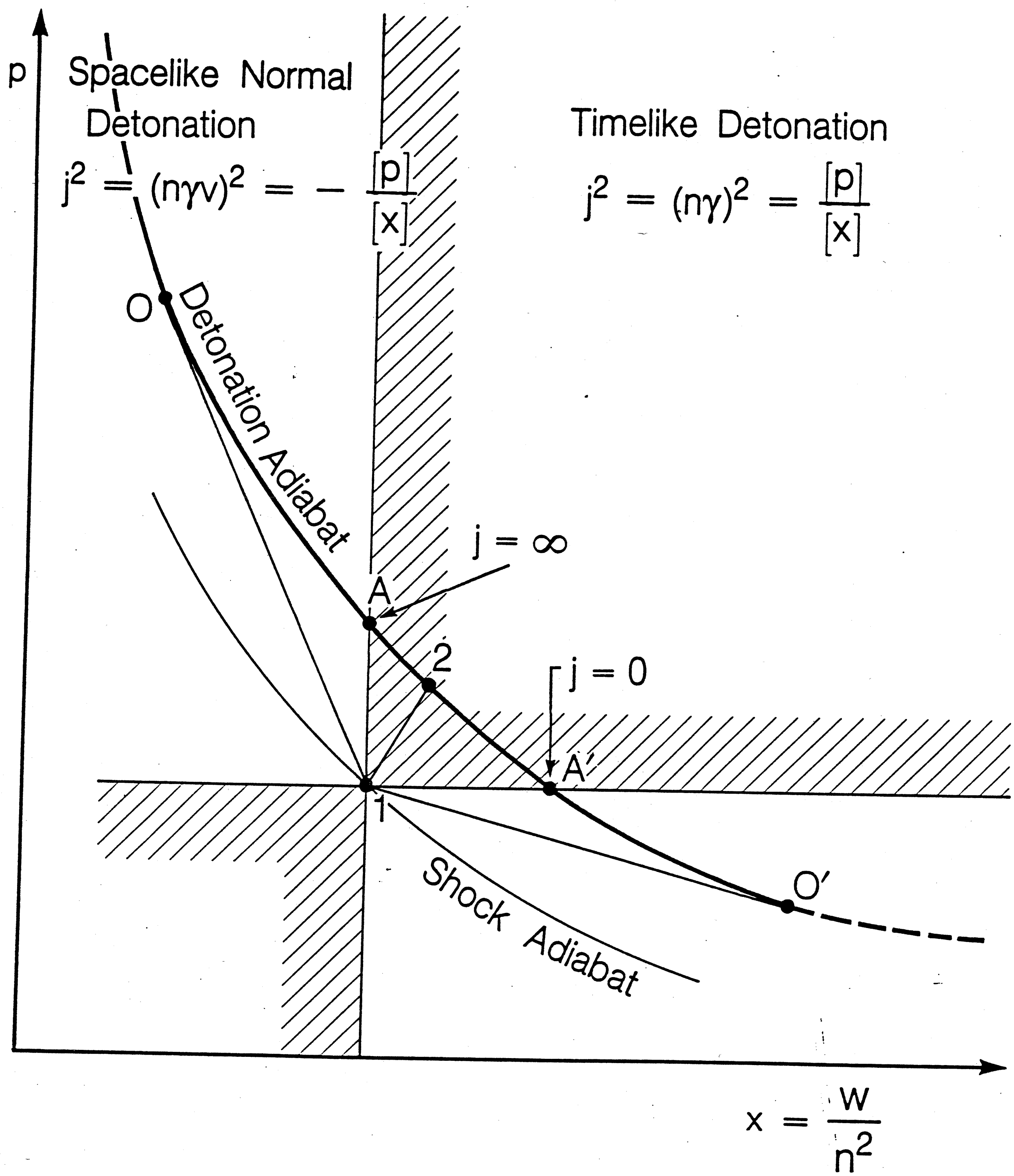

p
Spacelike Normal Detonation
$j^2 = (n\gamma v)^2 = -\frac{[p]}{[x]}$
Timelike Detonation
$j^2 = (n\gamma)^2 = \frac{[p]}{[x]}$
O
Detonation Adiabat
$j = \infty$
A
2
$j = 0$
A′
1
O′
Shock Adiabat
$x = \frac{w}{n^2}$

Fig. 2

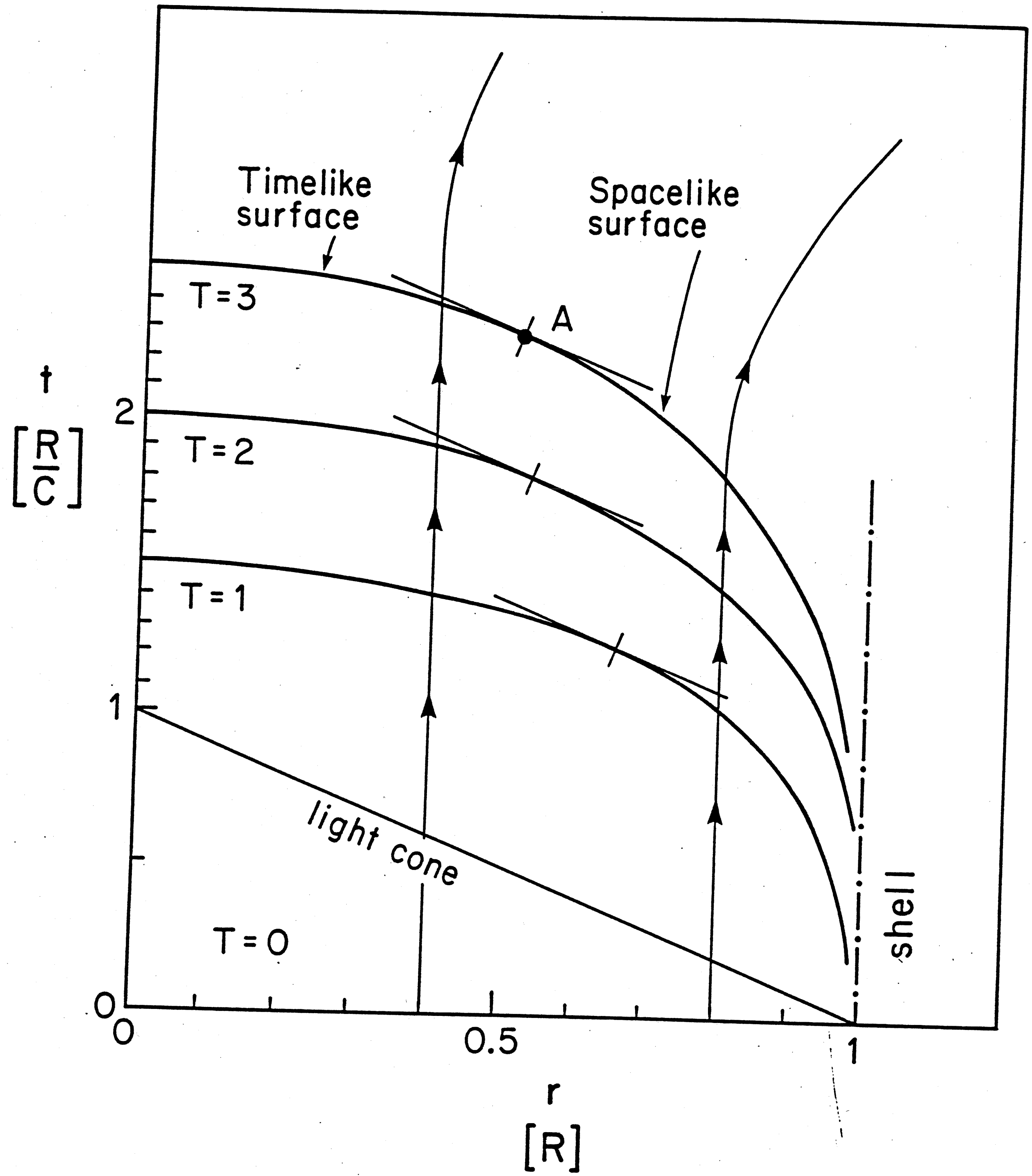

Timelike
surface
Spacelike
surface
T=3
A
T=2
T=1
T=0
light cone
shell
t
[R/C]
2
1
0
0
0.5
1
r
[R]

Fig. 3

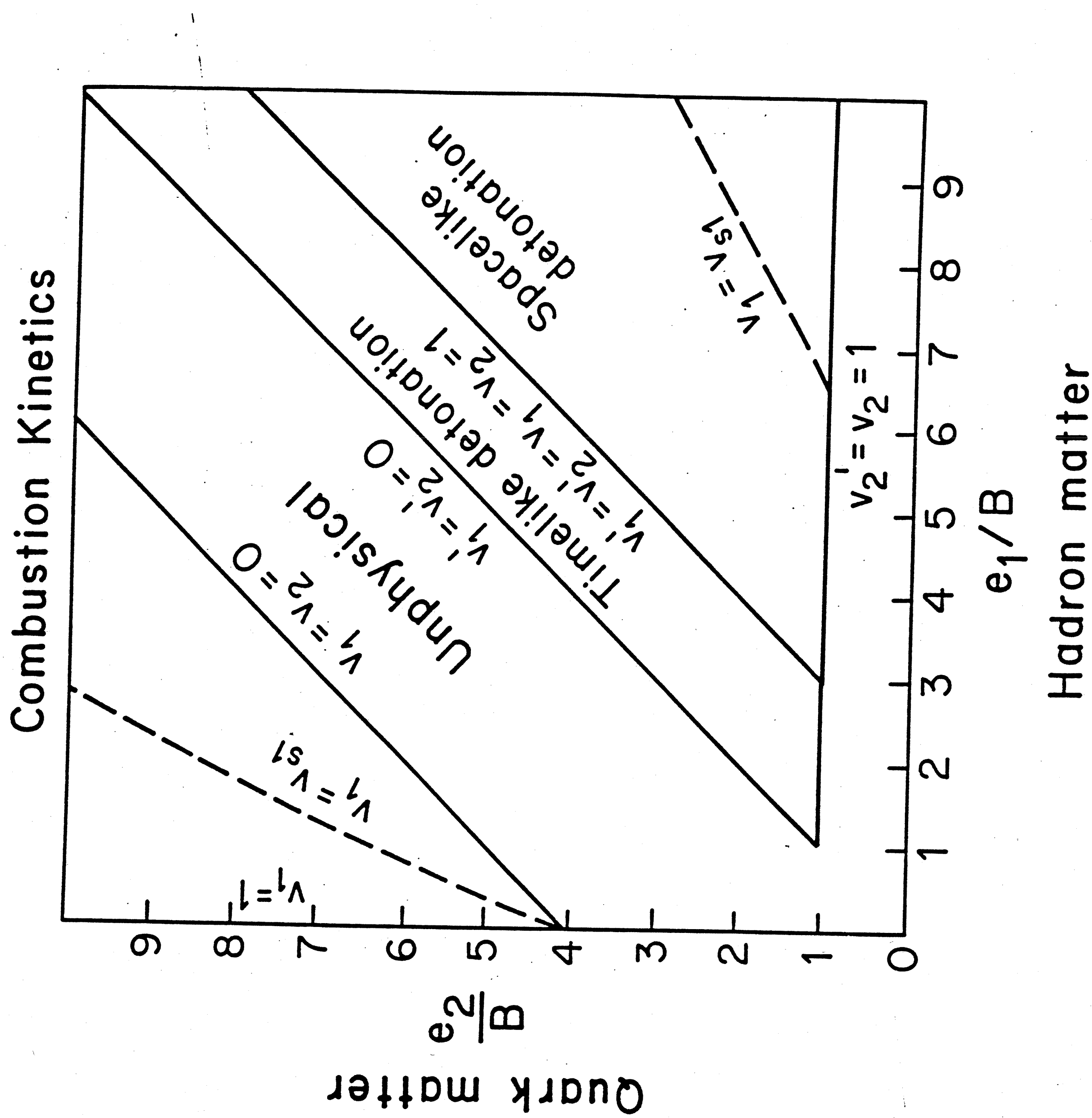

Combustion Kinetics
Spacelike detonation
Timelike detonation
Unphysical
v1 = vs1
v1 = vs1
v1' = v2' = v1 = v2 = 1
v1' = v2' = 0
v1 = v2 = 0
v1 = 1
v2' = v2 = 1
e1/B
Hadron matter
e2/B
Quark matter
0
1
2
3
4
5
6
7
8
9

Fig 4