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Abstract: The procedure of constraining the quantum dynamics to some particular trial manifolds is largely applied in order to obtain time-dependent approximations to the quantum evolution, and the connection between such results and the exact stationary states represents the requantization problem. In the following it will be shown that the requantization give also an explicit realization of the geometric objects encountered in the prequantization formalism, pointing out a direct correspondence between the stationary states and the reduced space for the invariant manifolds of the constrained flow. Moreover, it will appear that different techniques of the many-body theory as the random phase approximation and the method of the projection operators may receive an unified treatment within a general requantization formalism. This will operate a selection among the states appropriate for projection, and will allow to derive a form of generalized random phase approximation in terms of the representation operators for the semisimple Lie algebras, rather than of their explicit realizations using creation and annihilation operators.

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INTRODUCTION

The correspondence between the phase portrait of the classical dynamical systems and their quantum correspondent is a subject of increasing interest, motivated in part by the attempts to develop a consistent quantum description of the collective behaviour or of the chaotic dynamics [1].

The effective interactions for the realistic systems encountered in chemistry, nuclear physics or condensed matter physics are complicated, but reasonable approximations for the ground state may still be obtained if the dominant interaction term appears from a collective mean field. In such cases the low-lying spectra show a simple structure, described in terms of a small number of collective degrees of freedom. Their connection with the original quantum dynamics is a basic problem of the many-body theory, and a common procedure adopted for the separation of the collective subspace was first to construct an "artificial" classical system by constraining the quantum dynamics from the full Hilbert space $\mathcal{H}$ to some finite dimensional trial manifold, and than to search for slow decoupled modes [2-3]. Common examples of such constrained systems are the time dependent Hartree-Fock (TDHF) or Hartree- Fock - Bogolyubov (TDHFB) equations, which are derived for particular trial manifolds as the SU(n) or SO(2n) coherent states, respectively [4]. However the hybrid character of their solutions, including both quantum and classical aspects have given difficulties of interpretation. For an effective approximation to the exact quantum result it becomes therefore necessary to develop an additional "requantization" procedure, mapping the constrained dynamics to the original Hilbert space. This problem has not yet a complete solution, but were obtained important partial results. Thus, the methods of the induced representations ,[5], or of the geometric quantization ,[6], allow to overpass the difficulties dues to the peculiar geometry of the classical models, but the constrained dynamics is mapped in an abstract quantum Hilbert space ,constructed physically independent on the initial one. A different procedure ,applied in many recent works, consider the orbits dense on invariant tori of the constrained system as the key elements for requantization, and using the conditions of periodicity and "gauge invariance" (GIPQ), establishes a direct connection between them and the stationary states [7-9].

It is worth noting that if the classical phase space would be considered as a particular constraining manifold, than GIPQ would complement the relation between the periodic orbits and the density of states embodied in the Gutzwiller trace formula [10]. In this limit case the exact correspondence between the classical and the quantum dynamics is given by quantization rather than by requantization, and therefore one should expect a close relation between these two methods.

This paper represents an attempt to illustrate the connection between the GIPQ requantizaton of the constrained dynamics and the geometric prequantization method developed in order to construct representations for the Poisson algebra of the classical observables ,[11]. The problem of the constraining manifolds, known to play a crucial role, will be also discussed, considering instead of the HF or HFB manifolds general coherent states for semisimple Lie groups. In addition will be presented a procedure to build natural constraining manifolds on spontaneous symmetry breaking ground states ,[12].

The first part contains a description of the geometric structures associated with the exact quantum dynamics, as well as with the constrained dynamics on the trial function manifold.

In the second part is discussed the requantization of the integrable systems, pointing out the close analogy between prequantization and GIPQ. On examples it will be shown that GIPQ embodies naturally both the random phase approximation (RPA) used in the treatment of the small amplitude vibrations, and techniques developed for the restoration of the broken symmetries, as the method of the projection operators.

In the last part, the main results and conclusions are summarized.

I. The Constrained Quantum Dynamics

The full quantum dynamics in the Hilbert space $\mathcal{H}$ is given by the time-dependent Schrödinger equation ,(TDSE),:

$$i\hbar|\dot{\psi}\rangle = \hat{H}|\psi\rangle \tag{1}$$

where $\hat{H}$ is the Hamiltonian operator,defined on a dense domain in $\mathcal{H}$. This equation was related to the formalism of the classical mechanics by its derivation from the "variational principle" $\delta\mathcal{S}_{[\psi]}=0$, with $\mathcal{S}_{[\psi]} = \int dt\langle\psi|i\hbar\partial_t-\hat{H}|\psi\rangle$. However,it can also be seen as the equation corresponding to the flow of the quantum states $|\psi\rangle\in\mathcal{H}$,

$$|\dot{\psi}\rangle = X_H(\psi) \tag{2}$$

under the action of the Hamiltonian field $X_H(\psi)=-i/\hbar\hat{H}|\psi\rangle$ defined on $\mathcal{H}$ by the Hamilton function $h(\psi)=\langle\psi|\hat{H}|\psi\rangle$, $h:\mathcal{H}\longrightarrow\mathbb{R}$, and the symplectic form $\omega^{\mathcal{H}}\in\Omega^2(\mathcal{H})$ [13,4.2]:

$$\omega_{\psi}^{\mathcal{H}}(X,Y) = 2\hbar\ \mathrm{Im}\langle X(\psi)|Y(\psi)\rangle \tag{3}$$

$$X(\psi),Y(\psi)\in T_{\psi}\mathcal{H}$$

through the relation $i_{X_H}\omega^{\mathcal{H}}=dh$. Here and in the following $\mathcal{H}$ will be assumed finite dimensional ,although part of the results remain valid in the infinite dimensional case.

The invariance of h to the action of the group U(1) on $\mathcal{H}$, defined by $\phi_c|\psi\rangle=c|\psi\rangle$, $c\in U(1)$, lead to the conservation of the norm $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$,making possible the reduction of the dynamical problem from $\mathcal{H}$ to the projective space $\mathbb{P}_{\mathcal{H}}$ associated to $\mathcal{H}$ [14,5.5.C].

If $L = \{|\psi\rangle\in\mathcal{H}/\langle\psi|\psi\rangle=1\}$, and $\Pi:L\longrightarrow\mathbb{P}_{\mathcal{H}}$ is the projection :

$$\Pi(|\psi\rangle)\equiv[|\psi\rangle]=\{e^{i\varphi}|\psi\rangle,|\psi\rangle\in L,\varphi\in[0,2\pi]\} \tag{4}$$

then according to the general theory of the Marsden-Weinstein reduction , any curve $|Z_t\rangle^{\gamma}\in L$ having as projection $[|Z_t\rangle^{\gamma}]$, a solution γ_t of the reduced dynamical system defined on $\mathbb{P}_{\mathcal{H}}$, determines uniquely the TDSE solution on $\mathcal{H}$:

$$|\psi\rangle_t = \phi_{c_t}|Z_t\rangle^{\gamma} = e^{f(t)}|Z_t\rangle^{\gamma} \tag{5}$$

if

$$\dot{f}|Z_t\rangle^{\gamma} = X_{H(|Z_t\rangle^{\gamma})} - |\dot{Z}_t\rangle^{\gamma} \tag{6}$$

Assuming further $\hat{H}$ be time-independent, then $h=\langle\psi|\hat{H}|\psi\rangle$ is a constant γ and $|\psi\rangle_t$ becomes:

$$|\psi\rangle_t = e^{-\frac{i}{\hbar}\gamma t - \int_0^t d\tau\, {}^{\gamma}\langle Z_\tau|\partial_\tau|Z_\tau\rangle^{\gamma}}\,|Z_t\rangle^{\gamma} \qquad (7)$$

This result has a peculiar geometrical interpretation within the framework of the prequantization theory, because by the smooth , free and proper action $\phi: U(1)\times L \longrightarrow L$ of the "gauge group" U(1) ,L is a principal circle bundle over $\mathbb{P}_{\mathscr{H}}$ [14,4.1M]. However, each normed state in $\mathscr{H}$ generate a one dimensional Hilbert space ,and due to the natural action of the structure group on this space ,L appear also as the unit section in the associated complex vector bundle [15]. The 1-form $\alpha\in\Omega^1(L)$,$\alpha_\psi(X) = -i\hbar\langle\psi|X\rangle$,$X\in T_\psi L$,($d\alpha = \omega^{\mathscr{H}}|_L$), is a connection form on L, having as curvature the reduced symplectic form $\tilde{\omega}\in\Omega^2(\mathbb{P}_{\mathscr{H}})$, $\Pi^*\tilde{\omega} = \omega^{\mathscr{H}}|_L$.

Using the standard notations ,the covariant derivative of the states from L ,along γ on $\mathbb{P}_{\mathscr{H}}$,$\gamma_t = \Pi(|Z\rangle_t)$,is:

$$\frac{D|Z\rangle}{Dt} = \nabla_{\dot\gamma}\,|Z\rangle = \frac{i}{\hbar}\,\alpha(|\dot Z\rangle)\,|Z\rangle = \langle Z|\dot Z\rangle|Z\rangle \qquad (8)$$

Therefore, the curve:

$$|\tilde Z\rangle_t^{\gamma} = e^{-\int_0^t d\tau\, {}^{\gamma}\langle Z_\tau|\partial_\tau|Z_\tau\rangle^{\gamma}}\,|Z\rangle_t^{\gamma} \qquad (9)$$

is autoparallel along γ ,while $|\psi\rangle_t$ corresponds to the lift of the Hamiltonian current from $\mathbb{P}_{\mathscr{H}}$ to the bundle L.

Consider now instead L a trial manifold P of normed functions chosen as $M=\Pi(P)$ to be a symplectic manifold, with the symplectic form ω^M related to the restriction of $\omega^{\mathscr{H}}$ to P by $\Pi^*\omega^M$

$=\omega^{\mathscr{X}}|_{P}$. In this case denote by $h_M : M \longrightarrow \mathbb{R}$ the restriction of the Hamiltonian $h(|\psi\rangle)=\langle\psi|\hat{H}|\psi\rangle$ to M ,and by $\mathscr{X}\equiv(x^1,x^2,\dots x^{2N})$ both the parameters of the trial functions $|Z\rangle_{\mathscr{X}}\in P$, and the local coordinates of the point $\Pi(|Z\rangle_{\mathscr{X}})\in M$. Than the symplectic form ω^M is explicitly:

$$\omega^M = \sum_{i,j=1}^{2N} \omega_{ij} dx^i \wedge dx^j, \quad \omega_{ij}=\hbar \ \mathrm{Im}\langle\partial_i Z|\partial_j Z\rangle \qquad (10)$$

$(\partial_i \equiv \frac{\partial}{\partial x^i})$,and the constrained Hamiltonian current induced by h_M is given by the equations:

$$\sum_{j=1}^{2N} 2\ \dot{x}^j\ \omega_{ji} = \frac{\partial h_M}{\partial x^i} \qquad (11)$$

In addition if M is quantizable, so that $U(1)\cdot P$ is a principal bundle L_M over M ,than the original bundle structure $\Pi : L \longrightarrow \mathbb{P}_{\mathscr{X}}$. may be reproduced also for the constrained system. First of all, the restriction of the 1-form $\alpha\in\Omega^1(L)$ to L_M determines in the local system represented by the trial functions $|Z\rangle\in P$ the 1-form associated to the connection $\vartheta_Z\in\Omega^1(M)$:

$$\vartheta_Z = \sum_{k=1}^{2N} \langle Z|\partial_k|Z\rangle dx^k \qquad (12)$$

Than, along the lines of the prequantization theory, the connection make possible to lift the Hamiltonian flow from M to L_M. If $\mathscr{X}_t$ is an integral curve on M given by (11), its lift to L_M, denoted by $\hat{\rho}_t|Z\rangle$ is :

$$\hat{\rho}_t|Z\rangle \equiv \exp\left[-\frac{i}{\hbar}\, \mathscr{E}t + i\Theta_t \right] |Z\rangle_t \qquad (13)$$

$$\mathcal{H} = \langle Z|\hat{H}|Z\rangle \quad , \; |Z\rangle_t \equiv |Z\rangle_{\mathcal{X}_t} \; , \quad \Theta_t = \int_0^t d\tau \; {}_\tau\langle Z|i\partial_\tau|Z\rangle_\tau$$

and by construction, it gives the exact TDSE solution when P is extended to L. This function was obtained previously using the extended variational equation:

$$\delta\int\mathcal{L}(\mathcal{X},\dot{\mathcal{X}},r,\dot{r},\varphi,\dot{\varphi})\; dt = 0 \qquad (14)$$

with $\mathcal{L} = \langle\phi|i\hbar\partial_t - \hat{H}|\phi\rangle$ and $|\phi\rangle = re^{i\varphi}|Z\rangle_{\mathcal{X}}$, [7-8] and its similarity with the exact stationary solutions was the starting point for the GIPQ quantization. Here the presence of the additional variational parameters r,φ give the correct lift to L_M, showing that $\hat{\rho}_t|Z\rangle$ can be seen as the best approximation of the TDSE solutions in L_M.

The phase Θ_t is widely known as the Berry's phase, and it was derived first for the eigenfunctions of the adiabatic time-dependent Hamiltonians ,[16-19].

II. THE REQUANTIZATION OF THE INTEGRABLE SYSYEMS

In general the exact stationary states are not placed on the trial manifold P, but nevertheless they influence the phase portrait of the constrained system. The correspondence between the exact stationary states and the orbits of X_{h_M} on M will represent in the following the requantization problem for the time-dependent solutions, and as it was mentioned above ,a first result in this direction was the requantization of the closed orbits using the conditions of gauge invariance and periodicity (GIPQ) [7]. The GIPQ method was developed mainly for computational purposes, but its basic elements have also a simple geometrical interpretation within the prequantization formalism [11].

If $\mathscr{P}$ denotes the set of the closed orbits $\gamma_t = \Pi(|Z\rangle_t^\gamma)$ on M, than the GIPQ approximation to the stationary states is given by the autoparallel vectors:

$$|\tilde{Z}\rangle_t = e^{i\Theta_t} |Z\rangle_t^\gamma \;, \; \Theta_t = \int_0^t d\tau \; {}_\tau^\gamma\langle Z| i\partial_\tau |Z\rangle_\tau^\gamma \qquad ,\gamma\in\mathscr{P} \qquad (15)$$

which are periodic.

Using the "gauge" fixed by the functions $|Z\rangle^\gamma$ having the same period T_γ as γ ,denoted $|U\rangle^\gamma$, the periodicity condition $|\tilde{Z}\rangle_t^\gamma = |\tilde{Z}\rangle_{t+T_\gamma}^\gamma$ $,\gamma\in\mathscr{P}$, become a restriction on Θ, written as $\Theta_{T_\gamma} = 2k\pi, k\in\mathbb{Z}$.

Stated in terms of the 1-form ϑ_U this restriction is:

$$\frac{1}{2\pi i} \oint_\gamma \vartheta_U \in \mathbb{Z} \qquad (16)$$

but unlike the initial periodicity condition is gauge-dependent. Therefore, it is convenient to use the local relation $\omega^M = -i\hbar d\vartheta_U$ to express (16) in the intrinsic form:

$$\frac{1}{h} \int_\sigma \omega^M \in \mathbb{Z} \qquad ,\gamma\equiv\partial\sigma \qquad (17)$$

Consider now the dynamical system defined on $M=\Pi(P)$ to be completely integrable in the action-angle coordinates $(\mathfrak{p},\mathfrak{q})$, with $\mathfrak{q} \equiv (\varphi^1, \varphi^2, \ldots, \varphi^N)$ representing the angle coordinates on the torus T^N, and $\mathfrak{p} \equiv (I_1, I_2, \ldots, I_N) : M \longrightarrow \mathbb{R}^N$, N independent functions in involution representing the action variables. If there is no degeneracy, than the closed orbits correspond to the fundamental cycles $\{ \gamma_k, k=1,N \}$ of T^N , while any other orbit will cover densely T^N. According to [8] this particular feature of the general orbits should be also accounted by the quantization condition, proposing to extend the restriction (16) from the basic cycles to every closed curve γ on T^N. Consequently instead of selecting a discrete set of closed orbits for each fundamental cycle in part, one select a discrete set of invariant tori $\{T^N_k , k\in\mathbb{Z} \}$ requiring for all the action variables to be simultaneously integral multiples of $\hbar$.

This quantization of the action variables is the same as by the standard Bohr - Wilson - Sommerfeld condition, but it has also a direct correspondent within the prequantization formalism. To show this, let us recall that if (N,ω) is a presymplectic reducible manifold, and $\mathcal{F}$ is the foliation defined by the characteristic distribution of ω, than the reduced space $\overline{M}_0=N/\mathcal{F}$ is quantizable when :

$$\frac{1}{2\pi i} \oint_\gamma \vartheta \in \mathbb{Z} \tag{18}$$

with ϑ any global 1-form such that $\omega = d\vartheta$, and γ every closed curve contained in a leaf of $\mathcal{F}$,[20]. Now if the invariant torus T^N known to be a Lagrangean submanifold in M is considered as a limit case of presymplectic manifold, than $\overline{M}_0$ is a point, and the condition (16) in the extended form become the quantization condition for this particular reduced space. This approach has the quality of opening a new perspective on the requantization problem, because the appropriate correspondent for the stationary states appear to be the reduced space rather than the invariant torus.

As it was noticed before, although the set of autoparallel vectors in correspondence with the stationary states is limited by GIPQ only to those lying above an orbit γ_k^N dense on the quantized torus T_k^N, their dependence on time and initial conditions make them quite different on the exact eigenstates.

To avoid such difficulties, it was proposed further ,[8], to consider as an effective approximation for the exact eigenstates $|\psi_k>$ the time - average $|T_k^N>$ of the selected autoparallel vectors:

$$|T_k^N>= \lim_{t \longrightarrow \infty} \frac{1}{t} \int_0^t dt\ |\tilde{Z}>^{\gamma_k^N} \qquad (19)$$

This average is determined only by the quantized action variables, but the accuracy of this approximation may depend strongly on the choice of P. Therefore it is very important to analyse the results on particular examples by comparison with other methods, or to derive criteria for the selection of appropriate trial manifolds. Such examples will be provided in the following by the orbit cylinders and by the coherent states for semisimple Lie groups.

Example 1. *The exact orbit cylinders*

Let $\mathcal{P}= \left\{\gamma=\Pi(|Z>^{\gamma})/\ |Z>_t^{\gamma}\equiv e^{-\frac{i}{\hbar}\hat{H}t}|Z>_0^{\gamma}),\ t\in\mathbb{R}\right\}$ be a regular orbit cylinder in $\mathbb{P}_{\mathcal{H}}$ [14] corresponding to the exact TDSE solutions. The invariant tori are in this case the closed orbits $\gamma\in\mathcal{P}$. If γ is a periodic orbit, than $|Z\rangle^{\gamma}$ should be quasiperiodic, and the periodic gauge function $|U\rangle^{\gamma}$ can be written as "Bloch amplitude" $|U\rangle^{\gamma}_{t}= e^{i\varepsilon^{\gamma}t/\hbar}|Z\rangle^{\gamma}_{t}$. Here the quasienergy ε^{γ} is defined up to integer multiples of $\hbar\omega^{\gamma}= h/T_{\gamma}$, and is chosen within the first "Brillouin zone" $[-\hbar\omega^{\gamma}/2,\hbar\omega^{\gamma}/2]$. The quantization formula selecting the orbits γ^{n} now say ${}^{\gamma}_{o}\langle Z|\hat{H}|Z\rangle^{\gamma}_{o} = n\hbar\omega^{\gamma}+ \varepsilon^{\gamma}, n\in\mathbb{Z}$, and the associated stationary states:

$$|\gamma^{n}\rangle = \frac{1}{T_{\gamma^{n}}} \oint dt\, e^{-\frac{i}{\hbar}(\hat{H} - \mathcal{E}_{\gamma^{n}})t} |Z\rangle^{\gamma^{n}}_{o} \qquad (20)$$

represents the projection of the state $|Z\rangle^{\gamma}_{o}$ on the subspace of the eigenfunctions of $\hat{H}$ corresponding to the eigenvalue $\mathcal{E}_{\gamma^{n}}={}^{\gamma}_{o}\langle Z|\hat{H}|Z\rangle^{\gamma}_{o}$. The result is close to the ergodic mean theorem ,[14,3.7.24], and can be easily pictured for the harmonic oscillator, choosing the orbit cylinder associated to the Glauber coherent states.

For this simple system $\mathcal{H}$ is the Fock space ,and $\hat{H}=\hbar\omega(a^{+}a+ \frac{1}{2})$, with $a=\sqrt{(m\omega/2\hbar)}(\hat{x} + \frac{i}{m\omega}\hat{p})$. Let P be the Glauber manifold :

$$P=\{ |Z\rangle=\exp(za^{+} -z^{*}a)|0\rangle=e^{-|z|^{2}/2}\, e^{za^{+}}|0\rangle,\ z\in\mathbb{C}\} \qquad (21)$$

For this choice the trajectories on $M=\Pi(P)$ are given by $z_{t}= \exp(-i\omega t)z_{o}$, and the condition (16) fix the values $z^{n}_{o},\mathcal{E}_{n}$:

$$|z^{n}_{o}|^{2} = n \in \mathbb{N}$$
$$\mathcal{E}_{n} = \hbar\omega(|z^{n}_{o}|^{2} + \frac{1}{2}) = \hbar\omega(n + \frac{1}{2}) \qquad (22)$$

Moreover, ${}_t\langle Z|i\partial_t|Z\rangle_t = \omega\ |z_o|^2$, the period of the selected orbit γ^n is independent on n and the average of the function $|\tilde{Z}\rangle_t^{\gamma} = \exp(in\omega t + z_t^n a^+ - z_t^{n*} a)|0\rangle$, $z_t^n = e^{-i\omega t} z_o^n$, expressed by:

$$|\gamma^n\rangle = \frac{1}{T}\oint dt\ |\tilde{Z}\rangle_t^{\gamma^n} = \frac{1}{T}\oint\ e^{in\omega t - |z_o^n|^2/2}\ e^{z_t^n a^+}|0\rangle \qquad (23)$$

coincide up to a normalization factor with the exact solution $\sqrt{1/n!}(a^+)^n|0\rangle$.

It is worth noting that the "coherence" of the Glauber states make the lift $\hat{\rho}_t|Z\rangle$ be an exact TDSE solution.

Example 2. Coherent states for semisimple Lie groups

Consider $M = \Pi(P)$ to be the coherent state Kähler submanifold of $\mathbb{P}_{\mathcal{H}}$ generated by the projection of the group orbit $P=\{|Z\rangle = \hat{U}_a|\mathfrak{M}\rangle,\ a\in G,\ |\mathfrak{M}\rangle\in\mathcal{H},\ \mathfrak{M}\in \mathfrak{g}^*\}$ of the semisimple Lie group G through the highest weight vector $|\mathfrak{M}\rangle$ from $\mathcal{H}$ [21-22].

Choosing a local set of 2N real parameters $\{\ \xi_j,\quad j=1,2N\ \}$ and denoting by $\mathcal{D}_k\hat{U} \equiv \hat{U}^{-1}\partial_k\hat{U}$, $\hat{H}_U \equiv \hat{U}^{-1}\hat{H}\hat{U}$, than (10) give $2\omega_{jk} = i\hbar\langle\mathfrak{M}|[\mathcal{D}_j\hat{U},\mathcal{D}_k\hat{U}]|\mathfrak{M}\rangle$, and the equations (11) become:

$$i\hbar\sum_{j=1}^{2N}\dot{\xi}_j\langle\mathfrak{M}|[\mathcal{D}_j\hat{U},\mathcal{D}_k\hat{U}]|\mathfrak{M}\rangle = \langle\mathfrak{M}|[\hat{H}_U,\mathcal{D}_k\hat{U}]|\mathfrak{M}\ \rangle \qquad (24)$$

The global integration of these equations and the separation of the periodic orbits is a difficult task, but locally, around the critical points given by the minimum of $h_M = \langle\mathfrak{M}|\hat{U}^{-1}\hat{H}\hat{U}|\mathfrak{M}\rangle$,the periodic orbits exists , and may be found integrating the liniarized equations.

If $\Delta \in \mathfrak{g}^*$ is the set of roots for the Lie algebra $\mathfrak{g} \equiv T_e G$, with $\Sigma = \{\mathfrak{m} \in \Delta,\ (\mathfrak{m}, \mathfrak{M}) < 0\}$, and $\{\hat{E}_{\mathfrak{m}},\ \hat{H}_{\mathfrak{m}} / \hat{E}^+_{\mathfrak{m}} = \hat{E}_{-\mathfrak{m}},\ \hat{H}_{\mathfrak{m}} |\mathfrak{M}\rangle = (\mathfrak{m}, \mathfrak{M}) |\mathfrak{M}\rangle\}_{\mathfrak{m} \in \Delta}$ represents in $\mathcal{H}$ the Cartan-Weyl basis for $\mathfrak{g}$,than around a critical point $\hat{U}_o |\mathfrak{M}\rangle \in P$,the manifold P is parameterized by the real and imaginary parts of the complex variables $\{z_{\mathfrak{m}},\ \mathfrak{m} \in \Sigma\}$ defined through:

$$|Z\rangle = \hat{U}_z |\mathfrak{M}\rangle$$

$$\hat{U}_z = \hat{U}_o \exp \sum_{\mathfrak{m} \in \Sigma} (z_{\mathfrak{m}} \hat{E}_{\mathfrak{m}} - z^*_{\mathfrak{m}} \hat{E}_{-\mathfrak{m}}) \qquad (25)$$

In $\hat{U}_o |\mathfrak{M}\rangle$,for these parameters ω_{ij} become: $\omega_{x_{\mathfrak{m}}, x_{\mathfrak{n}}} = \omega_{y_{\mathfrak{m}}, y_{\mathfrak{n}}} = 0$,

$$\omega_{x_{\mathfrak{m}}, y_{\mathfrak{n}}} = -\hbar \langle \mathfrak{M} | [\mathcal{D}_{z_{\mathfrak{m}}} \hat{U}, \mathcal{D}_{z_{-\mathfrak{n}}} \hat{U}] | \mathfrak{M}\rangle = -\hbar\, \delta_{\mathfrak{m}\mathfrak{n}} \langle \mathfrak{M} | [\hat{E}_{\mathfrak{m}}, \hat{E}_{-\mathfrak{n}}] | \mathfrak{M}\rangle \qquad (26)$$

and retaining only the linear terms in z, the equations of motion are:

$$i\, \dot{z}_{\mathfrak{m}} \langle \mathfrak{M} | [\hat{E}_{\mathfrak{m}}, \hat{E}_{-\mathfrak{m}}] | \mathfrak{M}\rangle = \langle \mathfrak{M} | [\hat{H}_{U_o}, \hat{E}_{-\mathfrak{m}}] | \mathfrak{M}\rangle -$$

$$- \sum_{\mathfrak{n} \in \Sigma} \langle \mathfrak{M} | \Big[[(z_{\mathfrak{n}} \hat{E}_{\mathfrak{n}} - z^*_{\mathfrak{n}} \hat{E}_{-\mathfrak{n}}), \hat{H}_{U_o}], \hat{E}_{-\mathfrak{m}} \Big] | \mathfrak{M}\rangle \qquad (27)$$

The condition as $\hat{U}_o |\mathfrak{M}\rangle$ be a critical point is expressed by the equation :

$$\langle \mathfrak{M} | [\hat{H}_{U_o}, \hat{E}_{-\mathfrak{m}}] | \mathfrak{M}\rangle = 0 \qquad ,\forall \mathfrak{m} \in \Sigma \qquad (28)$$

determining $\hat{U}_o$. However, further will be taken $\hat{U}_o \equiv I$, supposing that if the critical minimum point exists, than it represents the ground state ,and defines the element $|\mathfrak{M}\rangle$ from P. In addition, this element will be assumed invariant to the symmetry group of $\hat{H}$. When the ground state has not this property, the action of the identity component for the symmetry group generate a critical manifold for h_M ,directly related to the occurrence of the collective modes. As important examples ,may be mentioned the anisotropic ground states within HF or the superfluid states within HFB. These are not invariant to the action of the rotation group SO(3,ℝ), respectively of the "gauge" group U(1) generated by the particle number operator.

Let $|\mathfrak{M}\rangle$ be a symmetry-preserving minimum point , and consider the periodic orbits associated to the normal vibration modes. These are found after the calculus of the complex amplitudes $\{X_{\mathfrak{m}}, Y_{\mathfrak{m}}, \mathfrak{m} \in \Sigma\}$ and of the oscillation period T occurring in the general periodic solution:

$$z^{\omega}_{\mathfrak{m}} = X_{\mathfrak{m}} e^{-i\omega t} + Y_{\mathfrak{m}} e^{i\omega t} \quad , \mathfrak{m} \in \Sigma, \omega = 2\pi/T \qquad (29)$$

Replacing this solution in the liniarized equation, a time-independent system is obtained:

$$\langle\mathfrak{M}| \Big[[\hat{H}, \hat{B}^{+}] - \omega \, \hat{B}^{+}, \, \hat{E}_{\mathfrak{m}} \Big] |\mathfrak{M}\rangle = 0 \quad , \forall \, \mathfrak{m} \in \Delta \ , (\mathfrak{m}, \mathfrak{M}) \neq 0 \qquad (30)$$

where $\hat{B}^{+}$ denotes the sum:

$$\hat{B}^{+} = \sum_{\mathfrak{m} \in \Sigma} (X_{\mathfrak{m}} \hat{E} - Y^{*}_{\mathfrak{m}} \, E_{-\mathfrak{m}}) \qquad (31)$$

For the periodic solution (29) the integrand in the formula (16) ,$\langle\mathfrak{M}|\hat{U}^{-1}\partial_{\tau}\hat{U}|\mathfrak{M}\rangle$, is dominated by the term:

$$\begin{aligned} & \sum_{\mathfrak{m} \in \Sigma} \langle\mathfrak{M}| [\hat{E}_{\mathfrak{m}}, \hat{E}_{-\mathfrak{m}}] |\mathfrak{M}\rangle \, \mathrm{Im}(\dot{z}_{\mathfrak{m}} z^{*}_{\mathfrak{m}}) = \\ & = -\omega \sum_{\mathfrak{m} \in \Sigma} \langle\mathfrak{M}| [\hat{E}_{\mathfrak{m}}, \hat{E}_{-\mathfrak{m}}] |\mathfrak{M}\rangle (|X_{\mathfrak{m}}|^{2} - |Y_{\mathfrak{m}}|^{2}) \end{aligned} \qquad (32)$$

and the quantization condition for the classical amplitudes becomes:

$$-\sum_{\mathfrak{m} \in \Sigma} \langle \mathfrak{M} | [\hat{E}_{\mathfrak{m}}, \hat{E}_{-\mathfrak{m}}] | \mathfrak{M} \rangle (|X_{\mathfrak{m}}|^2 - |Y_{\mathfrak{m}}|^2) = n \quad , n=1,2,\ldots \tag{33}$$

The result is meaningful only if the values of the quantized amplitudes are within the range allowed by the linear approximation employed. Assuming this to be true for n=1 ,the autoparallel section having the period T becomes:

$$|\tilde{Z}\rangle_t^{\omega} = e^{i\omega t} \exp(e^{-i\omega t} \hat{B}^+ - e^{i\omega t} \hat{B}) \, |\mathfrak{M}\rangle \tag{34}$$

and the stationary state given by the time average is:

$$|\omega\rangle = \frac{1}{T} \oint |\tilde{Z}\rangle_t^{\omega} \, dt = \hat{B}^+ |\mathfrak{M}\rangle \tag{35}$$

In the HF case, G=SU(n), and the set of generators associated to the roots $\mathfrak{m} \in \Sigma$ contains A(n-A) operators $\{c_i^+ c_j, A+1 \le i \le n, 1 \le j \le A\}$, defined using the single-particle fermion operators for creation and annihilation $\{c_i^+, c_i; i=1,n\}$.With these operators (30) becomes the RPA equation in the particle-hole channel ,and for the first excited state ,(n=1), the condition (33) reduces to the normalization equation for the RPA operators ,$\langle \mathfrak{M} | [\hat{B}, \hat{B}^+] | \mathfrak{M} \rangle = 1$.

Similarly, choosing G=SO(2n), the quantization of the normal vibrations lead to the RPA equations in the particle-particle channel.

Example 3. The symplectic manifolds induced by spontaneous symmetry breaking

Consider the symmetry group of the Hamiltonian $\hat{H}$ to be the N-dimensional torus G=U(1)×U(1)×...U(1), and assume that the ground state of $\hat{H}$ is non-invariant to the action of G given by the unitary representation operators :

$$\hat{U}_{q} = e^{-\frac{i}{\hbar}\sum_{k=1}^{N}\varphi^{k}\hat{L}_{k}} \qquad (36)$$

with $[\hat{L}_i, \hat{L}_k] = 0$ and $q \equiv (\varphi^1, \varphi^2, \dots, \varphi^N)$ the coordinates on G. Assume also that $\hat{L}_k$ are all time-even, namely by complex conjugation $\hat{L}_k^* = \hat{L}_k$, $k=1,N$.

This situation appears frequently for the approximate ground states obtained by constrained variational calculations, but may also be expected for the exact ones at the classical limit, when the system become macroscopic.Thus let (M,ω^M) be a general trial manifold invariant to G, and $|g\rangle$ a deformed ground state solution of the variational equation $\delta\langle Z|\hat{H}|Z\rangle=0$, with $|Z\rangle\in M$. This solution is not unique in general, and let assume further that all the minima with the lowest energy are on the critical submanifold $Q\subset M$,$Q=\{\hat{U}_{q(r)}|g\rangle, r\in G\}$ generated by the action of G on $|g\rangle$. Choosing the angles $(\varphi^1, \varphi^2, \dots, \varphi^N)$ as parameters on Q , it appears that $\omega^M_{\varphi^i\varphi^j}|_Q=0$, and therefore Q is an isotropic submanifold in M.

If Q is isotropic, than at any $q\in Q$ the tangent space T_qQ has a coisotropic ω^M-orthogonal complement F_q defined by $F_q=\{X\in T_qM|\omega^M(X,Y)= 0 \text{ for all } Y\in T_qQ\}$,[14], and the quotient $E_q = F_q/T_qQ$ is a symplectic vector space. Consider P_q to be a complement of F_q , that is $T_qM=P_q\oplus F_q$.Than P_q is isotropic, with the same dimension as $T_q Q$, and such that ω^M restricted to $P_q \times T_q Q$ is nondegenerate. Thus locally, at every point q of Q one has a classical phase space structure, with P_q representing the space of the action variables canonically conjugate to the angle coordinates $\mathfrak{q}$. The remaining variables may be considered as "intrinsic" and are represented within E_q.

Suppose now that at low energies the system has no intrinsic excitations and is completely integrable in action-angle coordinates. Than the angle coordinates will be clearly represented by the parameters of G, while the action coordinates will be represented by the components of the momentum mapping $\vec{J}: M \longrightarrow \mathbb{R}^N$, $J_{k(Z)} = \langle Z|\hat{L}_k|Z\rangle$ for the action of G on M [14].

Although this choice solve completely the classical problem, for requantization is necessary in addition to find explicitly an extension of Q to a symplectic submanifold $\mathscr{P}$ parameterized by $\mathfrak{q}$ and $\mathfrak{p}$. Let us assume that

$$\mathscr{P} = \left\{ |Z\rangle_{(\mathfrak{q},\mathfrak{p})} = \hat{U}_{\mathfrak{q}(r)} |Z\rangle_{(0,\mathfrak{p})},\ r \in G \right\} \qquad (37)$$

with $|Z\rangle_{(0,\mathfrak{p})}$ a vector depending smoothly on $\mathfrak{p}$, to be determined such that $\mathfrak{p}_k = \langle Z_{(0,\mathfrak{p})}|\hat{L}_k|Z_{(0,\mathfrak{p})}\rangle$, $|Z\rangle_{(0,\mathfrak{g})} \equiv |g\rangle$, and

$$\dot{\varphi}^k = \frac{\partial h_M}{\partial \mathfrak{p}_k} = \lambda^k \qquad (38)$$

Here $h_{M(\mathfrak{p},\mathfrak{q})} = h_{M(\mathfrak{p},0)} = \langle Z_{(0,\mathfrak{p})}|\hat{H}|Z_{(0,\mathfrak{p})}\rangle$, while $\vec{\lambda}$ and $\mathfrak{g}$ are constant vectors. A simple calculation shows that for this choice $2\omega^M_{\varphi^i p_k}|_{\varphi}=\delta_{ik}$, and $\omega^M_{\varphi^i\varphi^j}|_{\varphi}=0$.

According to (38), the vector $|Z_{(0,p)}\rangle$ may be defined as the minimum of h_M on $T^N=\mathcal{J}^{-1}(p)$ joined smoothly with $|g\rangle$. This constrained variational calculation becomes a normal one for the modified Hamiltonian $\hat{H}_c=\hat{H}-\sum_{k=}^{N}\lambda^k\hat{L}_k$,[23], intensively used within the self-consistent cranking model of the collective rotations. Thus, denoting by $|Z\rangle_\lambda$ the minima of $\langle Z|\hat{H}_c|Z\rangle$ and by $\vec{\lambda}_p$ the value determined from $p_k=\ _\lambda\langle Z|\hat{L}_k|Z\rangle_\lambda$, the result is $|Z\rangle_{(0,p)}=|Z\rangle_{\lambda_p}$. The time reversal properties of the operators $\hat{L}_k$ ensure than $\omega^M_{p_ip_k}|_{\varphi}=0$

This result complete the construction of the manifold P as well as of the invariant tori T^N parameterized by the action variables p. Considering the frequencies λ^k non-degenerate, the requantization procedure may continue along the standard lines with the selection of a discrete set of invariant tori and with the calculus of the approximate eigenstates of the Hamiltonian. As it was shown in [8] the time average of the orbits dense on a selected torus T^N may be replaced by the "space average", and finally the state (19) become:

$$|T^N\rangle = \frac{1}{2\pi} \prod_{k=1,N} \oint d\varphi^k e^{-\frac{i}{\hbar}(\hat{L}_k - p_k)\varphi^k} |Z\rangle_{(0,p)} \qquad (39)$$

representing the projection of the state $|Z\rangle_{(0,p)}$ on the common eigenvector of the operators $\hat{L}_k$, with the eigenvalues $p_k = n_k\hbar$, $n_k \in \mathbb{Z}$.

III. Discussion and Conclusions

The natural restriction of the quantum dynamics to the normed functions manifold was treated now as a model for quantizable constraining manifolds M not necessarily determined by symmetries.

Considering time-independent Hamiltonian systems, it was discussed the correspondence between the prequantization formalism and the GIPQ requantization of the constrained dynamics. For completely integrable systems this comparison has revealed the role of the reduced space associated to the quantized invariant tori as the appropriate correspondent to the stationary states. Therefore it becomes possible a direct extension to more complicated systems, when the reduced space projected from the invariant manifold is not a point.

The choice of the constraining manifold, known to play a central role for the accuracy of the requantization procedure, was also discussed, particularly important appearing the exact orbit cylinders or the manifolds related to the symmetry breaking critical points. Depending on this choice, the proposed method appear as a fine instrument either for the study of the exact quantum dynamics ,or of the collective modes. In particular,it shows the common nature of the problems solved usually by different techniques, as the projection methods and random phase approximation.

It is worth noting that among the harmonic oscillator constraining manifolds, the Glauber states are distinguished because include the ground state and the restricted dynamics becomes that of the associated classical states. Therefore, in this case the problem of quantization may be stated as the requantization of the constrained dynamics, rather than its correspondence with an isomorphic, but different quantum framework.

For the many-body systems an approximation of such coherent manifolds might be represented by the classical phase space induced by spontaneous symmetry breaking presented in the example 3, but in general remains an interesting open problem.

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