MAGNETIC FIELD DUE TO A CIRCULAR CURRENT

By

Bartin T. Smith

AN ABSTRACT

Submitted to the College of Science and Arts of
Michigan State University of Agriculture and
Applied Science in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

Department of Physics and Astronomy

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ABSTRACT

Mathematical expressions for the magnitude of the magnetic field due to a circular current are derived. The difficulty of evaluating elliptic integrals of modulus near 1 is obviated by successive applications of Landen's Transformation, resulting in expressions readily calculable on computers. The expressions for complete elliptic integrals of the first and second kinds are programmed, with explanations of order pairs. Directions for change of parameters are included. Programs for the magnitudes of the components of the magnetic field parallel and perpendicular to the axis are presented, with explanations as to scaling and use as a subroutine.

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I. INTRODUCTION

In many physical experiments it is desirable to know the magnitude at any location of a magnetic field due to a circular current. Though the mathematical expressions for the magnitudes are well known, the numerical difficulties involved in the evaluation of these expressions in any given case may be quite large.

It is the purpose of this thesis to consider such a case, and to show whereby the numerical difficulties may be alleviated. The mathematical operations necessary are readily adaptable to computer programming, and use has been made of the Michigan State University digital computer, Mistic.

The need for the ready calculation of magnetic fields due to circular currents arose in connection with the problem of current control in the trimming coils of the proposed Michigan State University cyclotron.

II. DERIVATION OF MAGNETIC FIELD MAGNITUDES

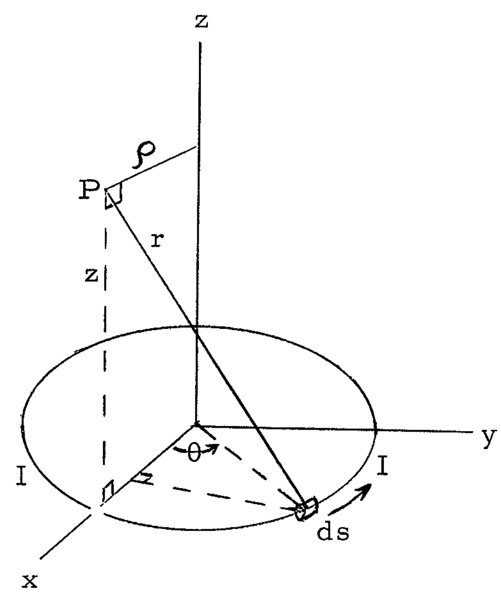

Fig. 1

We wish first to write an expression for the vector potential at any point P not on a coil of radius L. Noting that there is no loss of generality if we let P be in the x-z plane, we may write for the distance of P from any line element ds (see fig. 1):

$$r = \sqrt{L^2 + \rho^2 - 2L\rho\cos\theta + z^2} = \sqrt{(\rho - L\cos\theta)^2 + (L\sin\theta)^2 + z^2}$$

Using the expression for r, the vector potential is then:

$$\underline{A} = \frac{\mu_o I}{4\pi} \oint_c \frac{\underline{ds}}{\sqrt{(\rho - L\cos\theta)^2 + (L\sin\theta)^2 + z^2}}$$

When we consider opposing elements on opposite sides of the x axis, we see that there is no contribution from x components of the current, since the resultant current ($2I\cos\theta$) is perpendicular to the ρ - z plane.

Consequently, $A_\rho = A_z = 0$. So since $ds = a d\phi$,

$$A_\theta = \frac{\mu_o I}{2\pi} \int_o^\pi \frac{L \cos\theta d\theta}{\sqrt{(\rho - L\cos\theta)^2 + (L\sin\theta)^2 + z^2}}$$

In the manner of Smythe (5), let $\theta = \pi + 2\phi$, so $\cos\theta = 2\sin^2\phi - 1$, and $d\theta = 2d\phi$ so we have

$$A_\theta = \frac{\mu_o LI}{\pi} \int_o^{\frac{\pi}{2}} \frac{(2\sin^2\phi - 1)\, d\phi}{\left[\left\{(L+\rho)^2 + z^2\right\}\left(1 - \frac{4L\rho \sin^2\phi}{(L+\rho)^2 + z^2}\right)\right]^{\frac{1}{2}}}$$

Defining $k^2 \equiv \frac{4L\rho}{(L+\rho)^2 + z^2}$, and substituting, we get

$$A_\theta = \frac{\mu_o Ik}{2\pi}\left(\frac{L}{\rho}\right)^{\frac{1}{2}} \left[2\int_o^{\frac{\pi}{2}} \frac{\sin^2\phi\, d\phi}{[1-k^2\sin^2\phi]^{1/2}} - \int_o^{\frac{\pi}{2}} \frac{d\phi}{[1-k^2\sin^2\phi]^{1/2}}\right] \qquad [1]$$

Examining the first integral, we note that

$$\frac{\sin^2\phi}{[1-k^2\sin^2\phi]^{\frac{1}{2}}} = \frac{1}{k^2}\left(\frac{1}{[1-k^2\sin^2\phi]^{\frac{1}{2}}} - [1-k^2\sin^2\phi]^{\frac{1}{2}}\right)$$

so that

$$\int_o^{\frac{\pi}{2}} \frac{\sin^2\phi d\phi}{[1-k^2\sin^2\phi]^{\frac{1}{2}}} = \frac{1}{k^2}\int_o^{\frac{\pi}{2}} \frac{d\phi}{[1-k^2\sin^2\phi]^{\frac{1}{2}}} - \frac{1}{k^2}\int_o^{\frac{\pi}{2}} \left[1-k^2\sin^2\phi\right]^{\frac{1}{2}} d\phi = \frac{1}{k^2}(K-E)$$

where K and E are complete elliptic integrals of the first and second kinds, respectively.

The second integral of equation 1 is K, giving us a final expression for the vector potential:

$$A_\theta = \frac{\mu_o I}{\pi k} \left(\frac{L}{\rho}\right)^{\frac{1}{2}} \left[\left(1 - \frac{k^2}{2}\right) K - E\right]$$

The magnetic induction vector may be calculated from $\underline{B} = \nabla x \underline{A}$. Since there is no component of A in either the Z or ρ direction, this becomes

$$\underline{B} = -\hat{\rho} \frac{\partial A_\theta}{\partial z} + \frac{\hat{z}}{\rho} \frac{\partial(\rho A_\theta)}{\partial \rho}$$

$$\text{or } B_\rho = -\frac{\partial A_\theta}{\partial z} \text{ and } B_z = \frac{1}{\rho} \frac{\partial(\rho A_\theta)}{\partial \rho} \qquad [2]$$

Rewrite A_θ in the form:

$$A_\theta = \frac{\mu_o I}{\rho \pi} \left[(L+\rho)^2 + z^2\right]^{\frac{1}{2}} \left[\left(1 - \frac{2L\rho}{[(L+\rho)^2 + z^2]}\right) K - E\right]$$

$$\frac{\partial A_\theta}{\partial z} = \frac{\mu_o I}{2\rho\pi} z \left[(L+\rho)^2 + z^2\right]^{-\frac{1}{2}} \left[\left(1 - \frac{2L\rho}{[(L+\rho)^2 + z^2]}\right) K - E\right]$$

$$+ \frac{\mu_o I}{2\rho\pi} \left[(L+\rho)^2 + z^2\right]^{\frac{1}{2}} \left[\left(1 - \frac{2L\rho}{[(L+\rho)^2 + z^2]}\right) \frac{\partial K}{\partial k}\frac{\partial k}{\partial z} + K\left(\frac{4a\rho z}{[(L+\rho)^2 + z^2]}\right) - \left(\frac{\partial E}{\partial k}\frac{\partial k}{\partial z}\right)\right]$$

From Dwight, Table of Integrals (2):

$$\frac{\partial K}{\partial k} = \frac{E}{k(1-k^2)} - \frac{K}{k} \qquad \frac{\partial E}{\partial k} = \frac{E}{k} - \frac{K}{k}$$

Also, since $k = 2\left[\frac{L\rho}{(L+\rho)^2 + z^2}\right]^{\frac{1}{2}}$, we have $\frac{\partial k}{\partial z} = -\frac{2(L\rho)^{\frac{1}{2}} z}{[(L+\rho)^2 + z^2]^{\frac{3}{2}}}$

and $\frac{\partial k}{\partial \rho} = \frac{k}{2\rho} - \frac{k^3}{4\rho} - \frac{k^3}{4L}$

Substituting these expressions in Eq. [2] and simplifying, we obtain:

$$B_\rho = \frac{\mu_o I z}{2\rho\pi[(L+\rho)^2 + z^2]^{\frac{1}{2}}} \left[- K + \frac{L^2+\rho^2+z^2}{(L-\rho)^2 + z^2} E\right] \qquad [3]$$

For the z component, it can be shown similarly that

$$B_z = \frac{\mu_o I}{2\pi[(L+\rho)^2 + z^2]^{\frac{1}{2}}} \left[K + \frac{L^2-\rho^2-z^2}{(L-\rho)^2 + z^2} E\right] \qquad [4]$$

III. TRANSFORMATION OF ELLIPTIC INTEGRALS

Using the preceding formulae for B_ρ and B_z, one may calculate the modulus k and refer to tables for the corresponding K and E. However, since in a practical problem, k is not likely to be an even number with its complete elliptic integrals listed, it is necessary to interpolate between known values or to calculate them directly. With the availability of automatic calculating machinery, this latter course is preferable since it provides greater accuracy with a negligible sacrifice of speed.

In the regions where k is small, that is, when ρ is not near L or when z is large, the power series representation would be adequate, since its repetitive form lends itself well to computer calculation. When k is large, however, a situation that occurs at points near the coil, the series representation converges very slowly.

The power series is obtained by expanding the radical in a binomial series:

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = \int_0^{\frac{\pi}{2}} d\phi \left[1 + \frac{k^2}{2} \sin^2\phi + \frac{1\cdot 3}{2\cdot 4} k^4 \sin^4\phi + \dots \right.$$

$$\left. + \frac{1\cdot 3 \dots (2n-1)}{2\cdot 4 \dots (2n)} k^{2n} \sin^{2n}\phi + \dots \right]$$

Since by Wallis' Theorem: $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n}\phi d\phi + \frac{1\cdot 3 \dots (2n-1)}{2\cdot 4 \dots (2n)} \frac{\pi}{2}$

we have:

$$K = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \ldots + \left(\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}\right)^2 k^{2n} + \ldots\right]$$

As an illustration of this, consider the coefficient of the $n^{\underline{th}}$ term in the series for K:

$$\left[\frac{1\cdot 3\cdot 5\ \ldots\ (2n-1)}{2\cdot 4\cdot 6\ \ldots\ (2n)}\right]^2 = \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2$$

Using Stirling's approximation, $n! = \sqrt{2\pi n}\, n^n e^{-n}$, we may show that this fraction may be written $1/n\pi$. It can be seen that if this is the coefficient of k^{2n}, the result is a very slowly converging series when k is nearly 1.

That k can never be greater than 1 is true by definition. The modulus k is equal to the numerical eccentricity of an ellipse. For example, in calculating the arc length of an ellipse, the situation from which, according to Hancock (4), the "elliptic integral" derives its name, we have: if $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are the major and minor axes respectively,

$$S = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_0^x \sqrt{\frac{a^2 - \left(\frac{a^2 - b^2}{a^2}\right) x^2}{a^2 - x^2}}\ dx$$

Upon introducing the numerical eccentricity $e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2}$,

we have, with $x = a \sin\phi$,

$$S = a \int_0^\phi \sqrt{1 - e^2 \sin^2\phi}\ d\phi$$

which, it can be seen, is an elliptic integral of the second kind. Since $1 - \frac{b^2}{a^2}$ can never be greater than 1, the series will converge, however slowly. The quotient $\frac{b}{a}$ is called the complementary modulus k', and setting $e = k$, we have $k^2 + k'^2 = 1$.

A method of changing a slowly convergent elliptic integral into a highly convergent one is the method of Landen's transformation. Writing now $F(k, \phi)$ for the incomplete elliptic integral of the first kind, we wish to find a $F(k_1, \phi_1)$ such that it will be more rapidly convergent than $F(k, \phi)$ and can be related to it by $F(k, \phi) = C\, F(k_1, \phi_1)$, where C is a constant of proportionality to be determined.

Instead of using the radical $\sqrt{1 - k^2 \sin^2 \phi}$, substitute for the modulus k^2 the eccentricity $(a^2 - b^2)/a^2$, where, as before, $a > b$. This gives:

$$\sqrt{1 - k^2 \sin^2 \phi} = \frac{1}{a} \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}$$

and now: $F(k, \phi) = aF(a, b, \phi)$ and $E(k, \phi) = a^{-1} E(a, b, \phi)$

With this notation, we shall consider a geometrical derivation of Landen's transformation as given by Cayley (1).

On a circle of radius AOB, let P be any point, and Q be any point on the diameter other than the center, as in fig. 2. Considering a, b, and c_1 as noted in the figure, we may write

$a_1 = \frac{1}{2}(a+b)$, $b_1 = \sqrt{ab}$, and $c_1 = \frac{1}{2}(a - b)$ where a_1 is the radius, and $OQ = a_1 - b = \frac{1}{2}(a-b) = c_1$

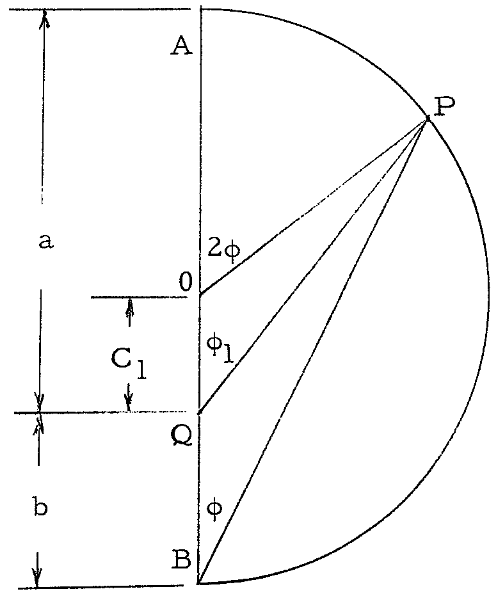

Fig. 2

We see that $QP \sin\phi_1 = a_1 \sin 2\phi$ and $QP \cos\phi_1 = c_1 + a_1 \cos 2\phi$.

Upon substituting the preceding expressions for a_1 and c_1, we obtain: $\overline{QP}^2 = a^2\cos^2\phi + b^2\sin^2\phi$ so that

$$\sin\phi_1 = \frac{a_1 \sin 2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}, \quad \cos\phi_1 \quad \frac{c_1 + a_1 \cos 2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin\phi}} \qquad [5]$$

and:

$$a_1^2\cos^2\phi + b_1^2\sin^2\phi = \frac{a_1^2(a\cos^2\phi + b\sin^2\phi)^2}{a^2\cos^2\phi + b^2\sin^2\phi}$$

We may write:

$$\sin(2\phi - \phi_1) = \sin 2\phi \cos\phi_1 - \cos 2\phi \sin\phi_1 = \frac{(a-b)}{2} \frac{\sin 2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}$$

and

$$\cos(2\phi-\phi_1) = \cos 2\phi \cos \phi_1 + \sin 2\phi \sin \phi_1$$

$$= \frac{a \cos^2\phi + b \sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}$$

$$= \frac{1}{a_1}\sqrt{a_1^2 \cos^2\phi_1 + b_1^2 \sin^2\phi_1}$$

Consider the point P' which is on the circle next to P. If ϕ_1 is incremented by $d\phi_1$, then:

$$PQd\phi_1 = PP' \sin P'PQ = PP' \cos(2\phi - \phi_1)$$

and since

$$PP' = a_1 d(2\phi) = 2a_1 d\phi$$

$$2 a_1 \cos(2\phi-\phi_1)\, d\phi = PQd\phi_1$$

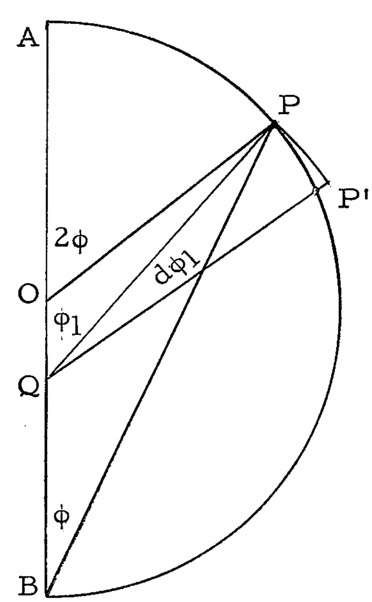

Fig. 3

Substituting for PQ and $\cos(2\phi - \phi_1)$ we get:

$$\frac{2d\phi}{a^2\cos^2\phi + b^2\sin^2\phi} = \frac{d\phi}{a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1}$$

Upon integrating we have

$$2\int_o^{\phi} \frac{d\phi}{a^2\cos^2\phi + b^2\sin^2\phi} = \int_o^{\phi_1} \frac{d\phi_1}{a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1}$$

or

$$F(a, b, \phi) = \frac{1}{2} F(a_1, b_1, \phi_1)$$

$$F(k, \phi) = aF(a, b, \phi) = \frac{a}{2} F(a_1 b_1 \phi_1) = \frac{a}{a+b} F(k_1 \phi_1)$$

which is the desired direction.

Writing $k_1 = \frac{1 - k'}{1 + k'} = \frac{a - b}{a + b}$, we have, since $1 + k_1 = \frac{2a}{a + b}$

$$F(k, \phi) = \frac{(1 + k_1)}{2} F(k_1, \phi_1)$$

When $\phi = \frac{\pi}{2}$, $\phi_1 = \pi$, and the complete integral is:

$$K(k, \frac{\pi}{2}) = \frac{1 + k_1}{2} F(k, \pi)$$

$$= \frac{(1 + k_1)}{2} \frac{(1 + k_2)}{2} F(k_1, 2\pi)$$

$$= (1 + k_1)(1 + k_2) \ldots (1 + k_n) \frac{F(k_n, 2^{n-1}\pi}{2^n}$$

where $$k_n = \frac{1 - k'_{n-1}}{1 + k'_{n-1}}$$

It will be recalled that a_1, b_1, and c_1 were derived from a, b, and c. Similarly a_2, b_2, and c_2 can be derived from a_1, b_1, and c_1.

This is equivalent to calculating the succeeding k_n.

$$a_1 = \frac{1}{2}(a+b) \qquad b_1 = \sqrt{ab} \qquad c_1 = \frac{1}{2}(a-b)$$

$$a_2 = \frac{1}{2}(a_1+b_1) \qquad b_2 = \sqrt{a_1 b_1} \qquad c_2 = \frac{1}{2}(a_1-b_1)$$

$$a_3 = \frac{1}{2}(a_2+b_2) \qquad b_3 = \sqrt{a_2 b_2} \qquad c_3 = \frac{1}{2}(a_2-b_2)$$

$$\vdots \qquad \vdots \qquad \vdots$$

It can be seen that as n increases, a_n and b_n approach the same limit. Hancock (3) shows that

$$a_1 - b_1 = \frac{(\sqrt{a} - \sqrt{b})^2}{2}$$

and

$$a_2 - b_2 = \frac{a_1+b_1}{2} - \sqrt{a_1 b_1} = \frac{a_1-b_1}{2} - (\sqrt{a_1} - \sqrt{b_1})\sqrt{b_1}$$

so

$$a_2 - b_2 < \frac{a_1-b_1}{2}, \text{ or } a_2 - b_2 < \frac{(\sqrt{a} - \sqrt{b})^2}{2^2}$$

Also we see that

$$a_3 - b_3 < \frac{a_2 - b_2}{2} < \frac{(\sqrt{a} - \sqrt{b})^2}{2^3}$$

and in general

$$a_n - b_n < \frac{(\sqrt{a} - \sqrt{b})^2}{2^n}, \text{ or } \lim_{n\to\infty} (a_n - b_n) = 0$$

When $a_n = b_n$, $k_n^2 = 1 - \frac{b_n^2}{a_n^2} = 0$, so that $\lim_{n\to\infty} F(k_n, 2^{n-1}\pi), = 2^{n-1}\pi$

and $K = (1 + k_1)(1 + k_2) \ldots\ldots (1 + k_n)\frac{\pi}{2}$

As a further illustration of the convergence of this series, we examine the log of the infinite product:

$$\ln K = \ln(1+k_1) + \ln(1+k_2) + \ln(1+k_3) + \ldots\ldots + \ln(1+k_n) + \ldots$$

When k_n is very small, $\ln(1+k_n) \approx k_n$, and the ratio of successive terms will be:

$$\frac{k_{n+1}}{k_n} = \frac{1-\sqrt{1-k_n^2}}{k_n(1+\sqrt{1-k_n^2})} = \frac{\frac{1}{k_n} - \sqrt{\frac{1}{k_n^2} - 1}}{1+\sqrt{1-k_n^2}}$$

Letting $k_n = \frac{1}{m}$, we see that

$$\lim_{m\to\infty} \frac{m-\sqrt{m^2-1}}{1+\sqrt{1-\frac{1}{m^2}}} = 0$$

Thus we see that continued application of this transformation will reduce an elliptic integral of any modulus k to a continuous product representation. Since the terms $(1+k_i)$ are calculated the same way, the method is readily adaptable to such machines as the Michigan State University digital computer, Mistic. In fact, the rapidity of convergence of the series for any k quickly exceeds the capacity of this machine, as it would any other. An example of the fast convergence is given in the following table:

k	0.999999 999990
k_1	0.999990 655984
k_2	0.991391 303175
k_3	0.768452 352529
k_4	0.219581 346615
k_5	0.012353 653210
k_6	0.000038 156098
k_7	0.000000 000365
k_8	0.000000 000000

In order to derive an expression for E in a more rapidly converging form, we begin with equation 5:

$$\sin \phi_1 = \frac{a_1 \sin 2\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}$$

Squaring, we get:

$$\sin^2 \phi_1 = \frac{4a_1^2 \sin^2 \phi \cos^2 \phi}{a^2 \cos^2 \phi + b^2 \sin^2 \phi}$$

Letting $D = a^2 \cos^2 \phi + b^2 \sin^2 \phi$ throughout the following, we write:

$$D - a^2 = -a^2 \sin^2 \phi + b^2 \sin^2 \phi = (b^2 - a^2) \sin^2 \phi$$

$$D - b^2 = (a^2 - b^2) \cos^2 \phi$$

$$(D - a^2)(D - b^2) = -(a^2 - b^2)(a^2 - b^2) \sin^2 \phi \cos^2 \phi$$

$$= -4a_1^2 (a-b)^2 \sin^2 \phi \cos^2 \phi$$

Since this expression is $[-(a-b)^2]$ times the numerator of the quotient above, we get: $(D-a^2)(D-b^2) + D(a-b)^2 \sin^2 \phi_1 = 0$.

Adding $[\frac{1}{2}(a^2+b^2)\cos^2\phi_1 + ab\sin^2\phi_1]$ to each side, we get, after much simplification:

$$D - [\frac{1}{2}(a^2+b^2)\cos^2\phi_1 + ab\sin^2\phi_1] = 4\, C_1^2\cos^2\phi_1(a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1)$$

Replacing D by its value, and rearranging, gives:

$$a^2\cos^2\phi + b^2\sin^2\phi = 2(a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1) - b_1^2 + 2C_1\cos\phi_1\sqrt{a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1}$$

Multiplying this equation by

$$\frac{d\phi}{\sqrt{a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1}} = \frac{\frac{1}{2}\,d\phi_1}{\sqrt{a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1}}$$

we get:

$$d\phi\sqrt{a^2\cos^2\phi + b^2\sin^2\phi} = d\phi_1\left[\sqrt{a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1} - \frac{\frac{1}{2}b_1^2}{\sqrt{a_1^2\cos^2\phi_1 + b_1^2\sin^2\phi_1}} + C_1\cos\phi_1\right]$$

Upon integrating, we obtain:

$$E(a, b, \phi) = E(a_1, b_1, \phi_1) - \frac{1}{2}b_1^2\, F(a_1, b_1, \phi_1) + C_1\sin\phi_1$$

From before, we had:

$$F(a, b, \phi) = \frac{1}{a}F(k, \phi), \quad E(a, b, \phi) = aE(k, \phi)$$

Using these expressions, we can write the transformed integral as:

$$E(k, \phi) = \frac{a_1}{a}E(k_1, \phi_1) - \frac{b_1^2}{2aa_1}\, F(k_1, \phi_1) + \frac{C_1}{a}\sin\phi_1$$

Continuing, however, with the integral in the form above:

$$E(a, b, \phi) = E(a_1, b_1, \phi_1) - \frac{b_1^2}{2}\, F(a_1, b_1, \phi_1) + C_1\sin\phi_1$$

we rewrite it as:

$$E(a, b, \phi) - a^2 F(a, b, \phi) = [E(a_1, b_1, \phi_1) - a_1^2 F(a_1, b_1, \phi_1)] - a_1 c_1 F(a_1, b_1, \phi_1) + c_1 \sin \phi_1$$

which, since $a_1^2 - \frac{a^2}{2} - \frac{b_1^2}{2} = -\frac{1}{4}(a^2 - b^2) = -a_1 c_1$, is equivalent to:

$$E(ab\phi) - a^2 F(ab\phi) = [E(a_1 b_1 \phi_1) - a_1^2 F(a_1 b_1 \phi_1)] - a_1 c_1 F(a_1 b_1 \phi_1) + C_1 \sin \phi_1$$

As n increases, we see that

$$\lim_{n \to \infty} [E(a_n, b_n, \phi_n) - a_n^2 F(a_n, b_n, \phi_n)] = 0$$

since $a_n = b_n$ means $k_n^2 = 1 - \frac{b_n^2}{a_n^2} = 0$

and $F(a_n, b_n \phi_n) = \frac{\phi n}{a_n}$, $E(a_n, b_n, \phi_n) = a_n \phi_n$

Now we can write, substituting for the expression in brackets its value obtained from the same equation with the subscripts increased by 1:

$$E(a, b, \phi) - a^2 F(a, b, \phi) = [E(a_2 b_2 \phi_2) - a_2^2 F(a_2 b_2 \phi_2)] - a_1 c_1 F(a_1 b_1 \phi)$$

$$+ c_1 \sin \phi_1 - a_2 c_2 F(a_2 b_2 \phi_2) + c_2 \sin \phi_2$$

$$= [E(a_2, b_2, \phi_2) - a_2^2 F(a_2, b_2, \phi_2)] - (2a_1 c_1 + 4a_2 c_2) F(a_2 b_2 \phi_2) + c_1 \sin \phi_1 + c_2 \sin \phi_2$$

As $n \to \infty$, we have:

$$E(a, b, \phi) - a^2 F(a, b, \phi) = -(2a_1 c_1 + 4a_2 c_2 + 8a_3 c_3 + \ldots) F(a, b, \phi) + c_1 \sin \phi_1 + \ldots$$

Since in our problem, $\phi = \frac{\pi}{2}$, $\phi_1 = \pi$, $\phi_2 = 2\pi$, ... , $\phi_n = 2^{n-1}\pi$,

$$E(a, b, \tfrac{\pi}{2}) = (a^2 - 2a_1 c_1 - 4a_2 c_2 - \ldots) F(a, b, \tfrac{\pi}{2})$$

but

$$\frac{1}{a}F(k, \phi) = F(a, b, \phi) = \frac{1}{2}F(a_1, b_1, \phi_1) = \ldots = \frac{1}{2^n} F(a_n, b_n, \phi_n)$$

and since $\phi_n = 2^{n-1}\pi$, we have: $\frac{1}{a}K = \frac{\pi}{2a_n}$

With this expression, and $E(k, \phi) = \frac{1}{a_n} E(a, b, \phi)$

we may restate $E(k, \frac{\pi}{2})$ as:

$$E(k, \frac{\pi}{2}) = \left[1 - \frac{2a_1c_1}{a^2} - \frac{4a_2c_2}{a^2} - \ldots\right] K(k, \frac{\pi}{2})$$

We see that $\frac{a_1c_1}{a^2} = \frac{a^2 - b^2}{4a^2} = \frac{k^2}{4}$ and $\frac{a_2c_2}{a_1c_1} = \frac{a_1^2 - b_1^2}{a^2 - b^2} = \frac{k_1}{4}$

so that $\frac{a_2c_2}{a^2} = \frac{a_1c_1}{a^2}\,\frac{a_2c_2}{a_1c_1} = \frac{k^2}{4}\,\frac{k_1}{4}$ and also $\frac{a_3c_3}{a_2c_2} = \frac{1}{4}k_2$

Succeeding terms are calculated in the same manner, allowing the final expression to be:

$$E = [1 - \frac{1}{2}k^2(1 + \frac{k_1}{2} + \frac{1}{4}k_1k_2 + \frac{1}{8}k_1k_2k_3 + \ldots)]K$$

Inasmuch as the k_n must be calculated in order to determine K, this form of E is easily evaluated without the necessity of complex programming, since each additional term of E consists of a number obtained from one multiplication and one right shift.

IV. PROGRAMS

A. Subroutine for E, K

The following is a program for the computing of K and E. This is a subroutine which is entered by a standard entry after the modulus k has been placed in the accumulator. K and E are then calculated in less than 40 milliseconds and placed in the accumulator and quotient register, respectively. A total of 59 memory locations are used to store this subroutine, and locations 0, 1, and 2 are used for temporary storage.

In the state in which the subroutine is presented here, the subroutine gives K and E scaled by 20, 10, or 5, but the scaling can be easily changed to a multiple of two by replacing the (. 2π) by π/4, π/8, etc., depending upon the range of the k to be used.

The square root subroutine from the Mistic library has been incorporated in this subroutine, and can be entered at (p+45), where the K, E subroutine begins at p. The standard entry is used and exit addresses will be computed normally.

As stated before, the representations for K and E to be programmed are:

$$K = (1+k_1)(1+k_2)(1+k_3) \ldots . (1+k_n)\frac{\pi}{2}$$

$$E = \left[1 - \frac{k^2}{2}\left(1 + \frac{k_1}{2} + \frac{k_1k_2}{4} + \frac{k_1k_2k_3}{8} + \ldots\right)\right] K$$

The K and E corresponding to sixteen randomly spaced k's were calculated on Mistic, the results agreeing to 10 places when $k < 0.999$ and to 8 places when $k \sim 0.9999$ with the values listed in Complete Elliptic Integrals, C. W. K. Gleerup, Lund University Press, 1957.

0	402F +5F		
1	4236L L5 2F		
2	4041L 41 38L		
3	4139L 4140L		
4	502F 7 J2F	k^2 (or k_n^2)	
5	102F 40F		
6	191F L0F		
7	40F 5 07L		
8	2645L 40F	$\sqrt{\frac{1}{4} - \frac{k^2}{4}}$	
9	LJF 401F		
10	L9F 661F		
11	-5F 402F	k_1 (or k_{n+1})	
12	101F 40F		
13	5038L 7JF	$\frac{k_1}{2} \frac{k_2}{2}$	This will be done on second passage. First passage multiplies by zero.
14	4038L L439L	$\frac{k_1}{2} + \frac{k_1}{2} \frac{k_2}{2}$	
15	4039L LJF	$\left(\frac{1}{2} + \frac{k_1}{2}\right)$	
16	401F 5040L		
17	7J1F 4040L	$\left(\frac{1}{2} + \frac{k_1}{2}\right) \left(\frac{1}{2} + \frac{k_2}{2}\right)$	

18	L537L 3222L	$(-2^{-38}) < 0$? For one passage only
19	L5F 4038L	
20	4039L LJF	
21	4040L L137L	Change sign of (-2^{-38})
22	4037L L542L	
23	L043L 4042L	Count for two jumps*
24	3626L L540L	
25	001F 4040L	$\left(\frac{1}{2}+\frac{k_1}{2}\right)\left(\frac{1}{2}+\frac{k_2}{2}\right)\left(\frac{1}{2}+\frac{k_3}{2}\right)\ 2$
26	L5F L037L	$\frac{k_n}{2} < 2^{-38}$
27	364L 5041L	
28	7J41L 101F	$\frac{k^2}{2}$
29	4041L 5039L	
30	7J41L L441L	
31	4041L 2659L	
32	LJ41L 4041L	
33	5040L7J44L	K
34	40F 5041L	
35	7JF 401F	E
36	2654L 22L	
37	LL4095FLL4094F	Test constant for size of $\frac{k_n}{2}$ (used as positive number)
38	00F 00F	
39	00F 00F	

*Scaling counter n must be picked such that $\frac{K}{2^{n-2}\pi} < 1$, eg.

$k > 0.99999$, use 3
$0.99999 > k > 0.9841$, use 2
$0.9841 > k > 0.802$, use 1

40	00F 00F	
41	00F 00F	
42	00F 002F	Scaling counter n
43	00F 001F	
44	40F 00 1283 1853 0718J	. 2π
45	401F+5F	
46	4253L511F	
47	101F-JF	
48	402F 50F	
49	L51F662F	R1 square root subroutine from computer tape library (45–53)
50	-5FL02F	
51	101F3653L	
52	L42F2648L	
53	L52F22F	
54	L557L 4037L	Restore counters (54–55)
55	L558L 4042L	
56	2661L 2661L	
57	LL4095F LL409 4F	Counter restoring constants (57–58)
58	00F 002F	
59	L941L 4039L	
60	LJ39L 2232L	
61	L5F 501F	
62	2236L 2236L	

B. Routine for $B\rho$

The routine for $B\rho$ is straightforward, and provisions have been made for any normalization the user may require.

The formula

$$B\rho = \frac{\mu_o I z}{2\rho\pi[(L+\rho)^2 + z^2]^{\frac{1}{2}}} \left[-K + \frac{L^2+\rho^2+z^2}{(L-\rho)^2+z^2} E \right]$$

has been changed with the aid of $\dfrac{k}{2\sqrt{L\rho}} = \dfrac{1}{\left[(L+\rho)^2+z^2\right]^{\frac{1}{2}}}$ and $\mu_o = (4\pi)10^{-7}$

to read:

$$\frac{B\rho}{I\ 10^{-7}} = \frac{kz}{\sqrt{L\rho}} \left[-K + \frac{L^2+\rho^2+z^2}{(L-\rho)^2+z^2} E \right]$$

Care must be exercised when this code is used because of the coefficient of E. When $\rho \approx L$ and Z is small, a large number results. For example, when $Z = \frac{1}{25} L$ and $\rho = L$,

$$\frac{L^2+\rho^2+z^2}{(L-\rho)^2+z^2} = \frac{2 + \frac{1}{625}}{\frac{1}{625}} = 1251$$

Clearly, the numerator must be scaled before the division takes place. A scaling factor of 10^{-m} has been used, with m determined by the user as appropriate for the particular location where the field is to be calculated. If ρ never approaches L, or if Z is large, then a smaller m would be sufficient, but if Z is small relative to ρ when ρ

is near L, then m will be large. The expression is then:

$$\frac{L^2+\rho^2+z^2}{(L-\rho)^2+z^2} = \frac{2L^2+z^2}{z^2}$$

Using this expression, an m can be chosen to fit the case at hand.

The first three words of this and the following program consist of carriage returns and delays to facilitate a change to a subroutine, should the user so desire.

Five random values of $B\rho$ were calculated on a desk calculator, the results agreeing with the Mistic calculated results to 9 places.

The routine for $B\rho$ will stop if the scaling factor is too large by hanging up at the division order at the 19th word.

0	92131F 923F	
1	92131F 923F	
2	92131F 923F	
3	L542L L443L	$(L+\rho)$
4	40F 50F	
5	7JF 40F	$(L+\rho)^2$
6	5044L 7J44L	z^2
7	L4F 40F	$(L+\rho)^2+z^2$
8	5042L 7J43L	$L\rho$
9	66F -5F	$\frac{L\rho}{(L+\rho)^2+z^2}$

10	40F 5010L	
11	26 KE + 45F 2650L	
12	-5F 4049L	Store E
13	L542L L043L	$(L-\rho)$
14	40F 50F	
15	7JF 40F	$(L-\rho)^2$
16	5044L 7J44L	z^2
17	L4F 40F	$(L-\rho)^2+z^2$
18	5049L 7J45L	
19	66F -5F	$\frac{E\times(\text{scale factor})}{(L-\rho)^2 + z^2}$
20	40F 5042L	L^2
21	7J42L 404F	
22	5043L 7J43L	
23	L44F 404F	$L^2+\rho^2$
24	5044L 7J44L	
25	L44F 404F	$L^2+\rho^2+z^2$
26	504F 7JF	$\frac{L^2+\rho^2+z^2}{(L-\rho)^2+z^2}$ (E) (scale factor)
27	40F 5048L	
28	7J45L L0F	[( )K - ( ) E]
29	405F 5042L	$L\rho$
30	7J43L 5030L	
31	26 KE + 45F 40F $\sqrt{L\rho}$	

32	50 54L 7J5F	
33	66F 7041L	$\frac{k[(\ \ )K - (\ \ )E]}{\sqrt{L\rho}}$ $\frac{zkB_{norm}[(\ \ )E - (\ \ )K]}{\sqrt{L\rho}} = B\rho$
34	6643L 7J44L	
35	52115F 5035L	Print
36	26P1F L547L	
37	L043L 3639L	Compare $\rho_{max} - \rho < 0$?
38	0FF0FF	
39	L543L L446L	Increase ρ
40	4043L262L	
41	B (norm)	
42	L (radius)	
43	ρ	
44	z	
45	scaling factor	
46	$\Delta\rho$	
47	ρ max	
48	00F 00F	
49	00F 00F	
50	00 1 F 4054L	k
51	4054L 5051L	Transfer to K, E
52	26K, EF 4048L	Store K
53	2612L 2612L	
54	00F00F	

C. Routine for B_z

As before with B_ρ, the routine for B_z was programmed in a straightforward manner, the only trouble being in calculating the coefficient of E.

The expression to be evaluated is, since $\mu_o = (4\pi) 10^{-7}$:

$$\frac{B_z}{I10^{-7}} = \frac{2}{\left[(L+\rho)^2 + z^2\right]^{\frac{1}{2}}} \left[K + \frac{L^2-\rho^2-z^2}{(L-\rho)^2+z^2} E\right]$$

Unlike the coefficient of E in B_ρ, $\rho = L$ presents little difficulty, as long as Z is not zero. When $Z \approx (1/25)L$, a scale factor of 10^{-1} is sufficient, since E is already scaled by 10 in the range $0.99999 > k > 0.9841$, and a smaller k would mean even less difficulty, since the coil would not be approached so closely.

When $Z \rightarrow 0$, however,

$$\frac{L^2-\rho^2-z^2}{(L-\rho)^2+z^2} \rightarrow \frac{L^2-\rho^2}{(L-\rho)^2} = \frac{L+\rho}{L-\rho}$$

Keeping $\frac{L+\rho}{L-\rho} < \frac{E}{10}$ (scaling factor) will suffice when $z << L$. For all but regions extremely near the coil, 10^{-1} or 10^{-2} would be ample scaling.

As with B_ρ, allowance has been made in the program for normalization with respect to any location the user might choose.

The values of B_z as calculated here agree to 9 places with those calculated by M. M. Mosely of Texas Christian University.

The routine for B_z will hang up if the proper scaling factor is not picked by failing to execute the divide order at the 28th word.

0	92131F 923F	
1	92131F 923F	
2	92131F 923F	
3	L542L L443L	
4	40F 50F	
5	7JF 40F	
6	5044L 7J44L	
7	L4F 409F	$(L+\rho)^2+z^2$
8	5042L 7J43L	
9	669F -5F	$\frac{L\rho}{(L+\rho)^2+z^2}$
10	40F 5010L	
11	26KE+45F 001F	k_1 (or k_n)
12	40F 5012L	
13	26KEF 4049L	Store K
14	-5F 4050L	Store E
15	L542L L043L	
16	40F 50F	
17	7JF 40F	
18	5044L 7J44L	
19	L4F 402F	$(L-\rho)^2+z^2$

20	5045L 7J50L	
21	401F 50 43L	
22	7J43L 40F	
23	5044L 7J44L	
24	L4F 40F	$L^2 - (\rho^2 + z^2)$
25	5042L 7J42L	
26	L0F 40F	
27	50F 751F	
28	662F -5F	$\frac{L^2 - (\rho^2 + z^2)}{(L-\rho)^2 + z^2}$ (E)(scaling factor)
29	408F 5049L	
30	7J45L L48F	[( )K + ( )E]
31	408F L59F	
32	L59F 5032L	
33	26KE+45F 409F	$\sqrt{(L+\rho)^2 + z^2}$
34	L58F 001F	
35	669F 7J48L	B_z
36	52115F 5036L	Print
37	26P1F L543L	
38	L046L 3641L	
39	L543L L447L	
40	4043L 262L	
41	OFF OFF	
42	L (radius)	

43	ρ	
44	Z	
45	scaling constant	
46	ρ max	
47	$\Delta\rho$	
48	$B_z(0)$	
49	00F 00F	Temporary storage for K and E
50	00F 00F	

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