THEORY -- OTHER

DOES SHORT-RANGE REPULSION MIMIC ANTISYMMETRY OF TWO-BODY WAVE FUNCTIONS?

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In many quantum-mechanical problems, an actual short-range potential interaction can be replaced by the zero-range forces. We consider a simple one-dimensional model of two identical particles in an infinite box with the residual interaction modeled by the δ-potential. Strangely enough, the complete solution of this problem is absent in the literature. Even with no interaction, the symmetry requirements for identical particles select the allowed two-body states. Although, for spatially symmetric states, the relative wave function does not vanish at short distances, it is suppressed by the repulsive δ-interaction. We solve the problem exactly and show that the quenching of the wave function becomes complete in the strong repulsion limit so that the *symmetric* wave function has the same energy and spatial stucture as the corresponding *antisymmetric* wave function of noninteracting particles when the short distances are eliminated by symmetry. In some sense, the δ-interaction mimics the change of the particle statistics.

The evolution of the lowest eigenvalues corresponding to the symmetric solutions is shown in Fig. 1. All eigenvalues behave qualitatively the same changing between two values which belong

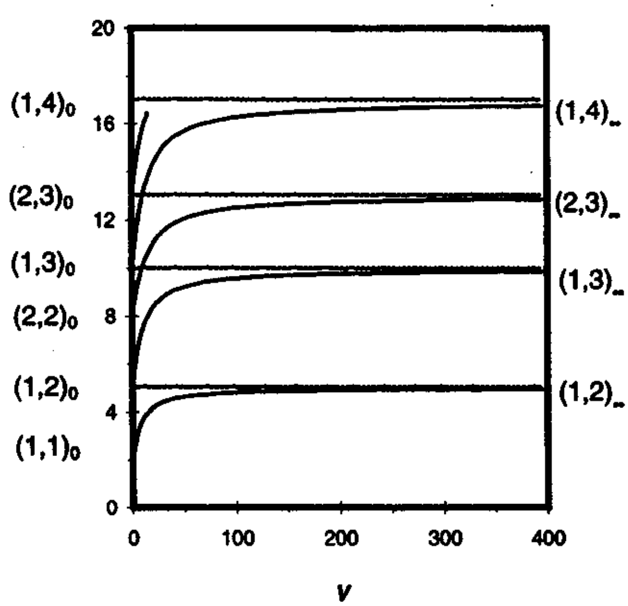

Figure 1: Energy spectrum of symmetric states of the two-particle system in a box with the delta-function repulsion as a function of the interaction strength v. The levels are labeled by the quantum numbers (n, n'), $n' \geq n$, of the occupied orbitals in the case of no interaction, $v = 0$. In the limit of $v \to \infty$, the terms come to the energies of unperturbed antisymmetric states $(n, n' + 1)$.

to the unperturbed spectrum. The energy terms $\epsilon_{nn'}(v)$, $n' \geq n$, start at the unperturbed values $\epsilon_{nn'} = (\pi/a)^2(n^2 + n'^2)$ corresponding to the noninteracting particles in the orbits n and n' in the square box of size a, then monotonically increase as a function of the repulsion strength v and saturate at the closest unperturbed values $\epsilon_{n,n'+1}$. The regularity of the pattern can be understood as follows. Consider for example the ground state $n = 1$, $n' = 1$. Its wave function $\Psi_{11} \propto \sin(\pi x/a)\sin(\pi y/a)$ has no nodes inside the box. The repulsive short range force suppresses the probability for the particles to be found at small $r = x - y$. The corresponding wave function for the interacting system (still continuous and positive everywhere) acquires the discontinuity of the derivative being proportional to $|r|$ near the diagonal line $r = 0$. In the limit of very strong repulsion v, the suppression becomes absolute, and the wave function goes to zero on the diagonal. Fig. 2 shows the spatial image and the contour map of the probabilities $|\Psi|^2$

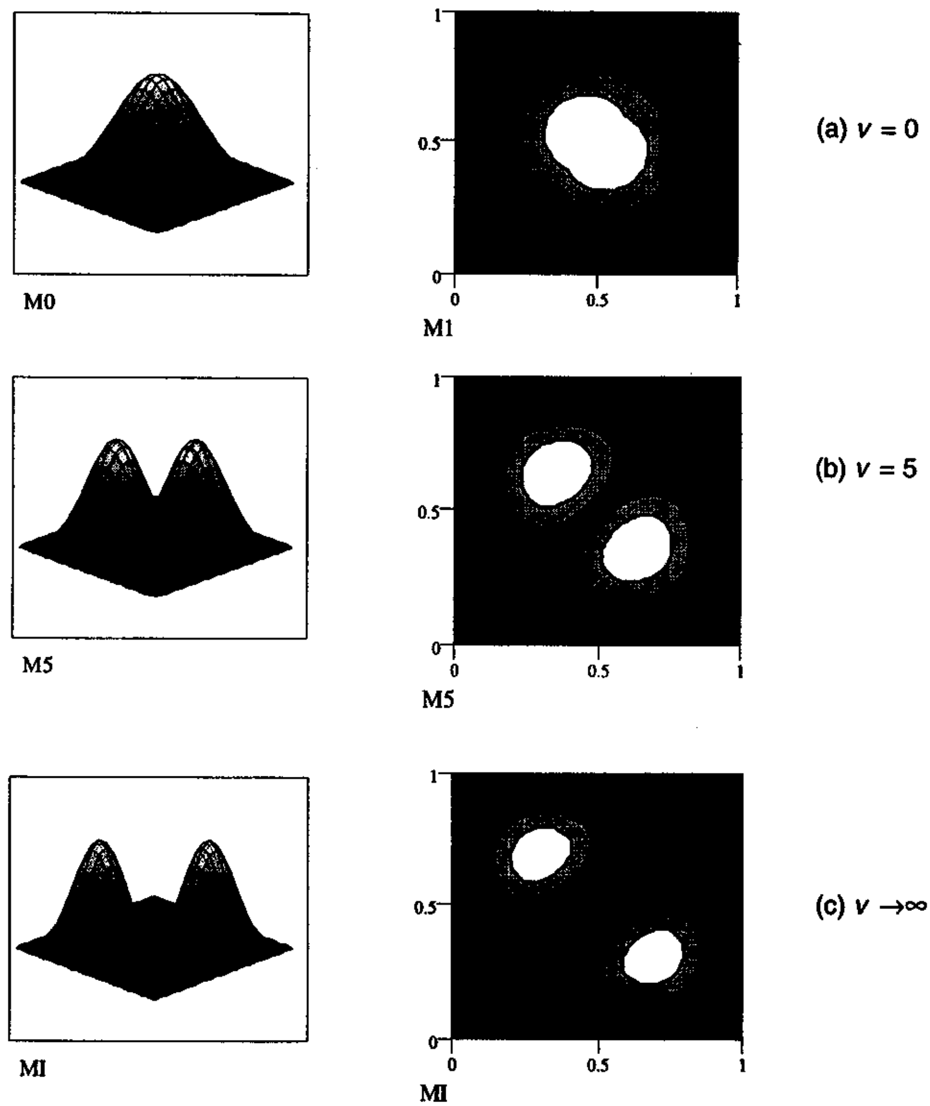

Figure 2: The spatial probability distribution, left column, and the corresponding contour maps, right column, for the lowest symmetric state ($n = n' = 1$ at $v = 0$) and for different values of the interaction strength, $v = 0$, part *(a)*; $v = 5$, part *(b)*, and $v \to \infty$, part *(c)*. The development of a deep valley at $r = x - y = 0$ is clearly seen.

for the ground state with no interaction, part *(a)*, at the intermediate interaction strength, part *(b)*, and in the limit of strong repulsion, part *(c)*.

It is easy to realize that the limiting function at very strong repulsion should coincide, up to mirror reflection, with the lowest antisymmetric noninteracting solution $\Psi_{12}^{(-)} \propto \sin(\pi x/a)\sin(2\pi y/a) - \sin(2\pi x/a)\sin(\pi y/a)$. Indeed, $\Psi_{12}^{(-)}$ vanishes, apart from the boundaries of the box, on the diagonal $r = 0$

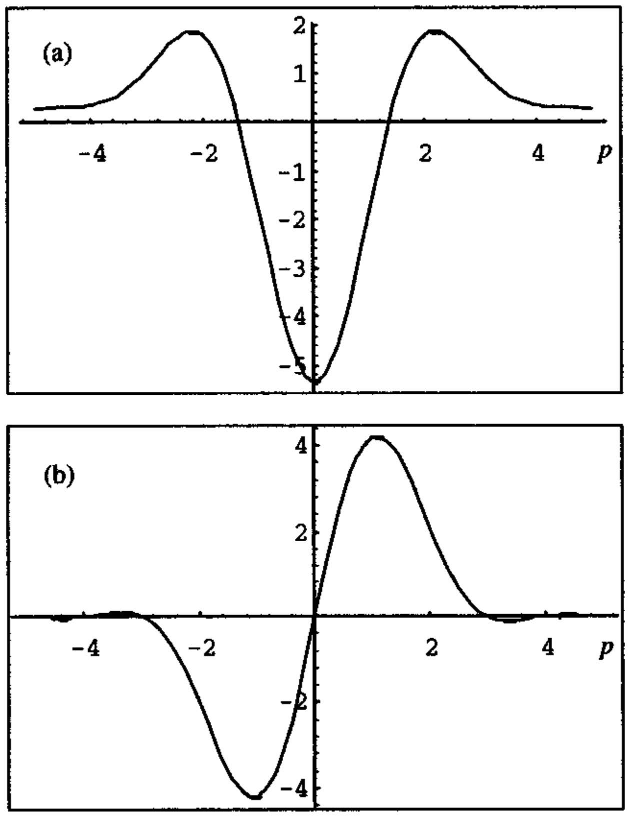

Figure 3: The wave functions $\phi(p)$ in the relative momentum representation, for the symmetric state $\Psi_{11}^{(+)}(v \to \infty)$, part *(a)*, and for the antisymmetric state $\Psi_{12}^{(-)}(v = 0)$ of the same energy, part *(b)*

where it remains continuous and changes its sign from negative in the triangle $-a \leq r < 0$ to positive in the triangle $0 < r \leq a$. Therefore $\Psi_{12}^{(-)}$, being nodeless inside each triangle, represents the lowest energy solution of the equation with zero conditions on the triangle boundaries. But the same is valid for the symmetric function at $v \to \infty$. The only difference is that the continuation of the symmetric function from the first triangle to the second one proceeds with mirror reflection which makes the function positive in both disconnected parts. Since, in the strong repulsion limit, the symmetric wave function vanishes on the diagonal, the energy eigenvalue and the probability distribution are identical with those for the lowest unperturbed antisymmetric solution. Similar consideration holds for higher states. There is a one-to-one correspondence between the starting symmetric solutions at $v = 0$, $\Psi_{nn'}^{(+)}(0)$, $n' \geq n$, and their descendants at $v = \infty$, $\Psi_{n,n'}^{(+)}(\infty)$, which have the same energy and the same probability distribution as the antisymmetric solutions $\Psi_{n,n'+1}^{(-)}(0)$ for $v = 0$. Thus, a strong repulsion at short distances makes the two-body spatially symmetric wave function look as belonging to the opposite symmetry, i.e. mimics the change of the statistics. The idea to include the Pauli antisymmetrization effects via short-range repulsion in a trial many-body wave function is quite natural and it was used repeatedly in nuclear physics. Our simple model partially justifies this procedure.

However, the wave function, in contrast to the spatial probability distribution, remembers its exact symmetry. The relative momentum distribution for the exact symmetric solution is different from that for the corresponding antisymmetric noninteracting solution. The relative momentum wave functions at the zero center-of-mass momentum ($p_x = -p_y = p$) are shown in Fig. 3 for the symmetric state (1,1) in the limit of $v \to \infty$ and for the antisymmetric state (1,2) at $v = 0$, *(a)* and *(b)*, respectively. The kink at the coinciding values of the coordinates is responsible for the appearance of the nonoscillating power law tail, $\propto p^{-2}$, in the symmetric wave function at large relative momenta. The symmetric solution has also a nonzero limit at zero relative momentum while the zero Fourier component of the antisymmetric function vanishes. Thus, the relative momentum wave function keeps memory of original particle statistics.

CASCADES IN WORD PROCESSORS AND SAND PILES: PREDICTABILITY AND REVERSIBILITY

Eric Eslinger, Scott Pratt and Wolfgang Bauer

Sandpiles, avalanches and the line-wrap feature in a word processor are all examples of cascading systems that are self-organized. The term "self organized" is inspired by the fact that the system maintains it's average conditions without external input, through the gradual build-up and sudden release of stress. Power law behavior, or correlations on all length scales, has been observed in many of these models, which has inspired them to be labeled as self-ordered critical (S.O.C.) systems[1,2]. Attempts have been made to understand such behavior in the context of an assortment of numerical and experimental examples, such as interface growth [3], sandpiles[4], numerical evolution models[5,6], earth quakes, and sliding blocks[7].

Here, we discuss two aspects of cascading systems: the distribution of strengths of catastrophic cascades and the the correlation of subsequent catastrophes. We have considered both word-processor line-wrap models[8] and sand-pile simulations, though in this report we focus on line-wrap simulations.

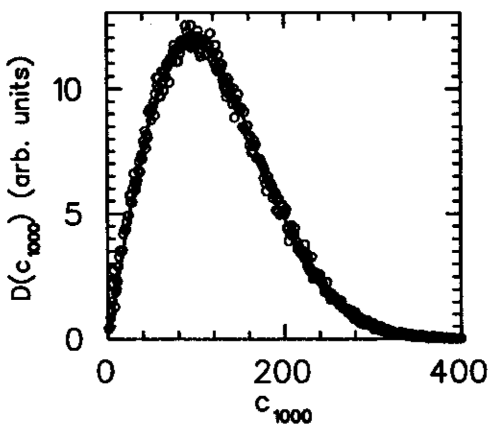

Figure 1: The distribution of line-wrap strengths shown above can be well described by comparison with a return-to-the-origin random problem or with the diffusion equation. The line is a fit using Eq. (1).

The line-wrap feature in a word processor is a particularly simple example. By entering words into the beginning of long paragraph, 1000 lines long for our model, line-wraps ensue. We label wraps which exceed the 1000 lines as catastrophes. The strength of the catastrophe is the number of characters c_{1000} that are pushed off the 1000th line. The distribution $D(c_{1000})$ is shown in Figure 1. Neglecting microscopic behavior, the distribution can be described by the formula:

$$D(c_{1000}) \propto c_{1000} \exp\left\{-\frac{c_{1000}^2}{2\eta}\right\} \tag{1}$$

This form can be understood by considering a random walk of 1000 steps, where the position of the random walk can be considered to be the number of characters pushed off the specific line in the paragraph. If the random walk returns to the origin, it dies. By analogy with the diffusion equation[8] one can show that the constant η in the equation above is proportional to the length of the paragraph and the variance of the word length distribution, which is can be thought of as the characteristic size of the random step.

The considerations above neglect the fact that stress is built up and released in the paragraph. Catastrophic cascades more likely occur when the stress is high, or when many characters n have been entered since the last catastrophe. We consider the probability $\Omega(n)$ of surviving n characters since the last catastrophe. The derivative, $d\Omega/dn$ is the probability of a catastrophe occurring n characters after the previous catastrophe.

$$\frac{d\Omega}{dn} = -\alpha(n)\Omega. \tag{2}$$

If there are no correlations, α is independent of n and Ω decays exponentially. The upper left panel in Figure 2 shows the probability of a catastrophe occurring at the n^{th} character. The right-hand side displays α, the conditional probability of a catastrophe given that one has survived thus far. It is zero at the origin and rises linearly. The fact that α rises with n is not surprising since stress is building in the

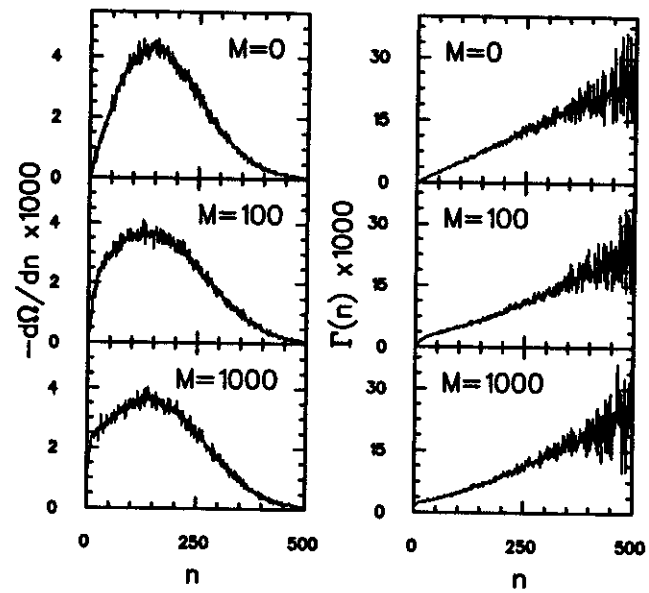

Figure 2: The distributions of characters entered between subsequent catastrophes is shown in the left-hand-side figures. The conditional probability α is shown in the right-hand-side figures. M pairwise switches of lines were performed after the entry of each word. For non-zero M (lower two panels), this destroys the reversibility of the system as shown by the non-zero intercept of the conditional probability.

paragraph, but the absolute disappearance at $n = 0$ is puzzling. This can be understood by noting that line-wrap simulations are reversible, that by erasing the first characters in the paragraph, backwards wraps ensue which exactly reproduce earlier states of the paragraph. This symmetry should manifest itself by requiring

$$\alpha(n) = -\alpha(-n). \tag{3}$$

To test this conjecture the middle and lower panel show the same simulations with the addition that M pairwise exchanges of lines are performed after the entry of each word. This destroys the reversibility while maintaining the aspect of building and releasing stress. The intercept is no longer zero after the destruction of the reversibility.

Dhar's sandpile algorithm[9] is also reversible, and it has been shown that this also leads to the same behavior[10]. Although the great majority of physical systems (e.g. earthquakes) are irreversible, this study demonstrates the role of dissipation in influencing the regularity and therefore the predictability of strong cascades.

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