PARTONS IN PHASE SPACE

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Primary hadronic collisions in a typical nuclear collision at RHIC will occur at $\sqrt{s} \sim 200A$ GeV. Such a collision is so violent that the partons, i.e. the quarks and gluons comprising the hadrons, will become deconfined. With hadronic densities exceeding the inverse volume of a typical hadron, the partons will remain deconfined, forming a quark-gluon plasma (QGP). Since transport theory descriptions of nuclear collisions have proven successful at lower energies [1], it is natural to attempt to describe the time evolution of the QGP using a QCD-inspired transport model. A transport model would describe the time evolution of the parton phase-space[1] densities throughout the collision. The procedures for deriving semiclassical transport equations using time-ordered nonequilibrium methods are well developed [1]. In fact, there have been several attempts at constructing a QCD transport model based on these procedures [2, 3, 4], but each of them have their problems. Chief among these problems is that one either treats the soft long-range phenomena (as in the case of [4]) or one treats the hard short-distance phenomena (in the case of [2]), but never both in the same framework. In each of these cases, the situation is chosen so that one has a clear separation between the length scale over which the particles travel (the kinetic scale) and the length scale of the interaction (the interaction scale). With this scale separation in hand, one can justify performing a gradient expansion. This expansion amounts to throwing out the short distance structure of the particle's phase-space density.

It many cases the gradient approximation is a good one. However, since its applicability depends on there being a separation of scales and that this scale separation only occurs in a limited set of situations in a RHIC collision, we can not be sure that the previous approaches to transport are valid. We have developed the beginnings of a technique that allows us to calculate the phase-space particle density *without* relying on the gradient expansion and so we do not need a scale separation for our calculations to work. The main tools in this investigation are the Generalized Fluctuation-Dissipation Theorem and phase-space propagators. We describe both in some detail and describe how one uses them to calculate particle densities. In the end, we use them to describe the shape of a parton cloud. This work is a summary of part of the work in Ref. [5].

We begin by defining what we mean by a phase-space particle density. First, we define the two-point functions $G^{><}(x,y)$ as expectation values of two field operators. For example for scalars we have:

$$\begin{aligned} iG^{>}(x,y) &= \left\langle \hat{\phi}(x)\hat{\phi}^*(y)\right\rangle, \\ iG^{<}(x,y) &= \left\langle \hat{\phi}^*(y)\hat{\phi}(x)\right\rangle. \end{aligned}$$

From here we define the particle densities through the Wigner transform of the $G^<$ functions:

$$f(x,p) = iG^{<}(x,p) = \int d^4(x-y)\; e^{i(x-y)\cdot p} iG^{<}(x,y) = \int d^4(x-y)\; e^{i(x-y)\cdot p} \left\langle \hat{\phi}^*(y)\hat{\phi}(x)\right\rangle. \tag{1}$$

We interpret $f(x,p)$ as the number density of particles (or antiparticles) per unit volume in phase-space

[1] By phase-space, we mean in space, time, momentum and energy (or invariant mass) simultaneously.

per unit invariant mass at time x_0:

$$f(x,p) = \frac{dn(x,p)}{d^3x\, d^3p\, dp^2}$$

We can see that this interpretation makes sense by integrating $f(x,p)$ over either x or p. When we integrate out p, we get the coordinate space particle density:

$$\int d^4p f(x,p) = \left\langle \hat{\phi}^*(x)\hat{\phi}(x) \right\rangle = \left\langle \hat{N}(x) \right\rangle ,$$

where $\hat{N}(x)$ is the number operator of the $\hat{\phi}$ field. The same way, when we integrate over x, we obtain the momentum space density. The off-shell Wigner function in Eq. (1) is related to the conventional Wigner function, $f_0(x,\vec{p})$, through the invariant mass integration:

$$f_0(x,\vec{p}) = \frac{dn(x_0,\vec{x},\vec{p})}{d^3x\, d^3p} = \int_{-\infty}^{\infty} dp^2 f(x,p).$$

This quantity is what one normally solves for in a transport model.

Now, to obtain equations of motion for $f(x,p)$ (or $f_0(x,\vec{p})$), one normally finds the equations of motion for $G^{><}(x,y)$ and then applies a Wigner transform. These equations of motion are known as the Kadanoff-Baym equations. The set of Kadanoff-Baym equations (one for each particle in a system) are non-perturbative as they contain the complete single particle information of the system. We can solve them formally, arriving at the Generalized Fluctuation-Dissipation Theorem. For the scalar particles, the Generalized Fluctuation-Dissipation Theorem reads as:

$$\begin{aligned} \mathrm{G}^{><}(1,1') = \; & \int_{t_0}^{\infty} d2 \int_{t_0}^{\infty} d3\; G^+(1,2)\Sigma^{><}(2,3)G^-(3,1') \\ & + \int d^3x_2 d^3x_3\; G^+(1,\vec{x}_2,t_0)G^{><}(\vec{x}_2,t_0,\vec{x}_3,t_0)G^+(\vec{x}_3,t_0,1'). \end{aligned} \tag{2}$$

The Generalized Fluctuation-Dissipation Theorem can be rewritten in phase-space by first taking $t_0 \to -\infty$ and Wigner transforming in the relative variable $x_1 - x_{1'}$. Doing so, we arrive at

$$G^{><}(x,p) = \int d^4y\, \frac{d^4q}{(2\pi)^4}\, \tilde{G}^+(x,p;y,q)\Sigma^{><}(y,q) + \lim_{y_0\to-\infty} \int d^3y\, \frac{d^4q}{(2\pi)^4}\, \tilde{G}^+(x,p;y,q)G^{><}(y,\vec{q}) \tag{3}$$

Here we recognize the Wigner transforms of the self-energy and initial particle density:

$$\Sigma^{><}(x,p) = \int d^4\tilde{x}\, e^{ip\cdot\tilde{x}}\Sigma^{><}(x+\tilde{x}/2, x-\tilde{x}/2) \tag{4}$$

and

$$\delta(t_0 - x_0)\, G^{><}(x,\vec{p}) = \int d^4\tilde{x}\; e^{ip\cdot\tilde{x}}\delta(t_0 - (x_0 + \tilde{x}_0/2))\,\delta(t_0 - (x_0 - \tilde{x}_0/2))\, G^{><}(x+\tilde{x}/2 x - \tilde{x}/2). \tag{5}$$

The delta functions render the initial density independent of p_0. We have also defined the retarded propagator in phase-space:

$$\tilde{G}^{+}(x,p;y,q) = \int d^4x'\, d^4y'\; e^{i(p\cdot x' - q\cdot y')} G^{+}(x+x'/2, y+y'/2)\, G^{-}(x-x'/2, y-y'/2)\,. \tag{6}$$

In phase-space, the Generalized Fluctuation-Dissipation Theorem takes a simple meaning. The phase-space self-energy $\Sigma^{><}(y,q)$ gives the quasi-probability density[2] for creating or destroying a particle with momentum q at point y. The full phase-space propagator, $\tilde{G}^{+}(x,p;y,q)$, describes how this particle propagates from y with momentum q to x with momentum p.

To solve Eq. (3), one must make some kind of approximation. Usually one applies the gradient approximation to Eq. (3), eliminating the d^4x' integral. We do not do this; instead, we assume the translational invariance of the advanced and retarded propagators. This is reasonable at lowest order in the coupling since the free field advanced and retarded propagators *are* translationally invariant. Making this approximation, the retarded propagator becomes

$$\begin{aligned} \tilde{G}^{+}(x,p;y,q) &= (2\pi)^4\delta^4(p-q)\int d^4z\; e^{ip\cdot z} G^{+}(x-y+z/2)\left(G^{+}(x-y-z/2)\right)^{*} \\ &\equiv (2\pi)^4\delta^4(p-q)\, G^{+}(x-y,p)\,. \end{aligned} \tag{7}$$

We will use $G^{+}(x-y,p)$ in all subsequent calculations. As an additional approximation, we only use the lowest order contribution to $G^{+}(x-y,p)$. This means that we dress the $><$ propagators but not the $\pm$ propagators when we iterate Eq. (3). Thus, our particles propagate as though they are in the vacuum. In the next few paragraphs we calculate the lowest order contribution to $G^{+}(x-y,p)$. With this propagator, the phase-space Generalized Fluctuation-Dissipation Theorem (Eq. (3)) simplifies to:

$$G^{><}(x,p) = \int d^4y\, G^{+}(x-y,p)\,\Sigma^{><}(y,p) + \lim_{y_0\to-\infty}\int d^3y\, G^{+}(x-y,p)\, G^{><}(y,\vec{p}) \tag{8}$$

In terms of the particle densities, we have

$$f(x,p) = \int d^4y\, G^{+}(x-y,p)\, i\Sigma^{<}(y,p) + \lim_{y_0\to-\infty}\int d^3y\, G^{+}(x-y,p)\, f_i(y,\vec{p}). \tag{9}$$

Here, f_i is the initial particle phase-space density. All of this discussion is done with scalar fields; the generalization to spinor and vector fields is straightforward and contained in [5].

[2] Because the self-energy is defined through a Wigner transform, it is not positive definite. Nevertheless, if we were to smooth it over unit volumes of phase-space, we could render it positive definite. It is the smoothed version of the self-energy that can be truly called a probability.

Next, we want to describe how particles propagate in phase-space. Given the definition of $G^+(x,p)$ in Eq. (6), we can perform the integrals with the lowest order (free-field) retarded propagator, for massless particles, analytically. We find the scalar retarded propagator in phase-space:

$$G^+(\Delta x, q) = \frac{1}{\pi}\theta(\Delta x_0)\theta(\Delta x^2)\theta(\lambda^2)\frac{\sin\left(2\sqrt{\lambda^2}\right)}{\sqrt{\lambda^2}}.$$

In this expression, the Lorentz invariant λ^2 is $\lambda^2 = (\Delta x \cdot q)^2 - q^2\Delta x^2$.

This propagator has several notable features. First, we see a $\theta(x_0 - y_0)$ which enforces causality and a $\theta((x-y)^2)$ which keeps the particle inside the light-cone. The rest of the features are tied in the dependence on the Lorentz invariant λ^2. First, we note that $\sin(2\sqrt{\lambda^2})/\sqrt{\lambda^2}$ gives rise to damped oscillations (the Wigner oscillations). As a scale for the drop off in $\sin(2\sqrt{\lambda^2})/\sqrt{\lambda^2}$, we take $\sqrt{\lambda^2} \leq 1$. This then gives us estimates for limits on propagation distance. With $q_\mu = (q_0, q_L, \vec{0}_T) = q_0(1, v_L, \vec{0}_T)$, for space-like momentum λ^2 becomes

$$\lambda^2 = q_0^2((x_0 - y_0)v_L - (x_L - y_L))^2 + q^2(\vec{x}_T - \vec{y}_T)^2 \leq 1$$

This tell us that if $q^2 = 0$, the particle follows its classical trajectory, i.e. $(x_0 - y_0)v_L + y_L = x_L$, with deviations $\mathcal{O}(1/q_0)$. Furthermore, if $((x_0 - y_0)v_L - (x_L - y_L))^2 = 0$, the particle still can propagate to distances of $\mathcal{O}\left(1/\sqrt{|q^2|}\right)$ transverse to its classical trajectory. This suggests the picture shown in Fig 1. Classically, a particle should follow a straight line trajectory $\vec{y} = \vec{v}\Delta t + \vec{x}$ as indicated in the figure. However, since the particle is quantum mechanical, it can end up anywhere within some region surrounding $\vec{y}$. This region is an ellipsoid with transverse width given by the invariant mass of the particle $1/\sqrt{-q^2}$ and longitudinal width given by the energy of the particle $1/q_0$. By comparison, under the gradient approximation all particles follow straight line classical trajectories, ignoring the effects of true quantum propagation.

We would like to use our insight from Eq. (9) to discuss the shape of the parton distribution of a nucleon. We begin by confining the valence quarks in nucleon bag with radius R_{bag}. For the parton model to make sense, this nucleon must be moving close to c, Lorentz contracting it in the direction of travel. The parton cloud of a nucleon is thought to arise from the fluctuation of a valence quark into a valence quark plus gluon(s). Since the gluon(s) can radiate other gluons or can virtually split into a quark-antiquark pair(s), we must account for all such processes. This is most conveniently done in the Leading Logarithm Approximation in which we sum up classes of diagrams such as is shown in Fig. 2. Feynman diagrams of this type can be written in phase-space and in the figure we outline how the different parts of the diagram correspond to the Generalized Fluctuation-Dissipation Theorem.

Now, as one travels down the ladder, one requires certain orderings of the parton momentum to pick up the leading contribution (the Leading Logs) to the process. This momentum ordering tells us the momentum of the last (n^{th}) parton in the ladder. For both partons with large q^2 (the DGLAP partons) and partons with small x (the BFKL partons)[3], the momentum of the n^{th} parton is such that the partons can not travel far from their creation points. So, knowing the rough momentum dependence as we go down the ladder, we can guess the shape of the i^{th} generation from the shape of the $i-1^{th}$ generation [5]; the i^{th} generation's distribution is only marginally larger than the $i-1^{th}$ generation[4].

[3] x is the fraction of the nucleon's longitudinal momentum carried by the parton

[4] Recoil from the radiated parton, corresponding to the cut rung, plays only a small role in the spreading from one generation to the next.

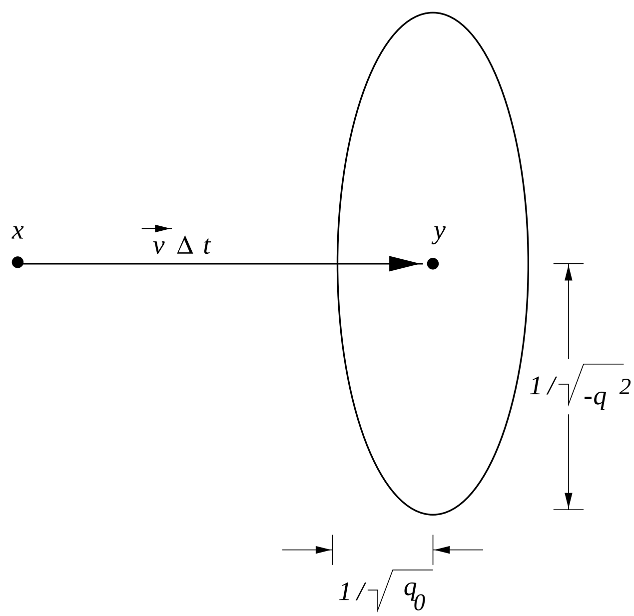

Figure 1: Classically, the particle propagates from $\vec{x}$ to $\vec{y}$. Quantum mechanically, the particle can propagate to anywhere within some region dictated by the particle's momentum.

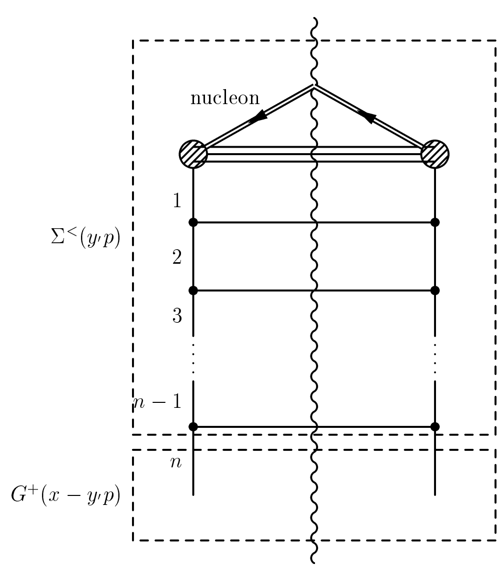

Figure 2: Cut diagram for the parton density in the Leading Log. Approximation. The upper part of the diagram corresponds to the parton source $\Sigma^{<}(y,p)$ while the bottom part, consisting of two dangling parton lines, corresponds to the phase-space propagator $G^{+}(x-y,p)$.

For the DGLAP partons, we require both the invariant mass and the longitudinal momentum fraction to be ordered as we go down the ladder [6, 7]: $-q_n^2 \gg -q_{n-1}^2 \gg \ldots \gg -q_1^2 \gg 1/R_{bag}^2 \approx \Lambda_{QCD}^2$ and $1 \geq x_1 \geq \ldots \geq x_{n-1} \geq x_n$. Furthermore, since $q_0 \approx q_L \approx xP_L \gg q_T, \sqrt{-q^2}$, the retarded propagator sends the n^{th} parton out to $R_{n\perp} \sim \hbar c/\sqrt{-q_n^2}$ transverse to the parton momentum and to $R_{n||} \sim \hbar c/q_{n0} = \hbar c/x_n P_L$ parallel to the the parton momentum. Thus, the n^{th} parton's distribution has a width of $\Delta R_T \sim R_{bag}$ in the transverse direction and a width of $\Delta R_L \sim R_{bag}/\gamma + \hbar c/xP_L$ in the longitudinal direction. This is illustrated on the left in Fig. 3. This $1/x$ dependence of the spread in the longitudinal direction is exactly what Mueller finds solely based on uncertainty principle based arguments [8].

For BFKL partons, we require an ordering in x only [7]: $1 \gg x_1 \gg \ldots \gg x_{n-1} \gg x_n$. BFKL-type evolution has only a weak dependence on the virtuality of the partons as we move down the ladder, so we assume q^2 to be fixed: $q_{n-1}^2 \approx q_n^2 \gg 1/R_{bag}^2$. Since x is small, the momentum of each parton can be predominantly transverse. In fact, the magnitude of q_T varies wildly because the particle momentum undergoes a random walk in $\ln(q_T^2)$ [7]. This is a well known effect of iterating the BFKL kernel (equivalent to moving down the ladder). Only in the case where the transverse momentum is small do we get an appreciable spread in the parton distribution. In this case, we find that the width in the transverse direction is again $\Delta R_T \sim R_{bag}$. However, the longitudinal extent, which is given by $\Delta R_L \sim R_{bag}/\gamma + \hbar c/|q_0|$, varies from $\Delta R_L \sim R_{bag}/\gamma + \hbar c/\sqrt{-q^2} \ll R_{bag}$ to $\Delta R_L \sim R_{bag}/\gamma + \hbar c/\sqrt{q^2 + (xP_L)^2 + q_T^2} \gg R_{bag}$, depending on whether q_T is large or small compared to the longitudinal momentum. Now, since the partons are all space-like, the BFKL partons can have a spread that is significantly larger than the nucleon they belong to. This observation is the basis for treating the whole nucleus as a sheet of fluctuating color charge in the McLerran-Venugopalan model [9].

In conclusion, we have developed the beginnings of perturbation theory in phase-space. Central to this theory is the phase-space Generalized Fluctuation-Dissipation Theorem which describes how a particle is created and propagates in phase-space. We discussed in some detail how particles propagate in phase-space. Because this can be done without making the gradient approximation and this does not rely on a separation of length scales, our work could eventually be used to test the validity of standard applications of transport theory. Finally, we discussed the shape of the cloud of DGLAP and BFKL partons. We found the size of DGLAP parton distribution to be moderately wider than the underlying bag, consistent with the uncertainty-principal based arguments of [8]. We also found that the shape of BFKL cloud is large in the longitudinal direction, consistent with expectations based on the model of McLerran and Venugopalan [9].

References

1. P. Danielewicz, Ann. Phys. **152** 239 (1984); S. Mrowczynski and P. Danielewicz, Nucl. Phys. B **342**, 345 (1990); S. Mrowczynski and U. Heinz, Ann. Phys. **229**, 1 (1994); B. Bezzerides and D. F. DuBois, Ann. Phys. **70**, 10 (1972).
2. K. Geiger, Phys. Rev. D **54**, 949 (1996).
3. P. A. Henning, "Procedings of the 4^{th} International Workshop on Thermal Field Theories," Dalian, China (1995); P.A.Henning, E.Quack and P.Zhuang, GSI preprint GSI-96-57 (1996) (submitted to Phys. Rev. D).
4. J. P. Blaizot and E. Iancu, Nucl. Phys. B **417**, 608 (1994).
5. D. A. Brown and P. Danielewicz, nucl-th/9802015.
6. G. Altarelli and G. Parisi, Nucl. Phys. B **126**, 298 (1977); C. Quigg. *Gauge Theories of the Strong, Weak and Electromagnetic Interactions.* Addison-Wesley, New York (1983).

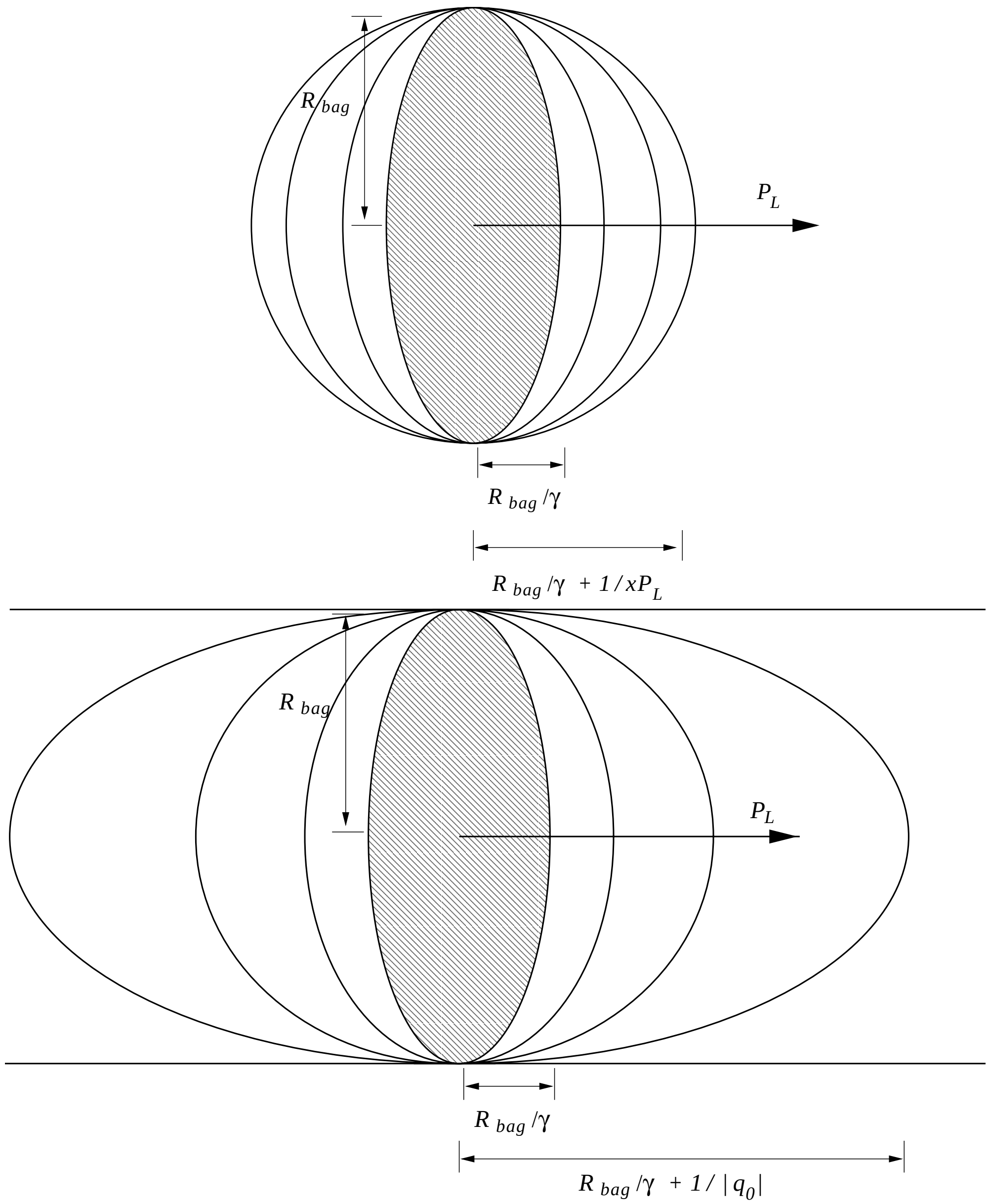

Figure 3: On the left: the shape of the DGLAP parton cloud. On the right: the shape of the BFKL parton cloud.

7. E. Laenen and E. Levin, Ann. Rev. of Nucl. Part. Sci. **44**, 199 (1994).
8. A. H. Mueller, Nucl. Phys. A **498**, 41c (1989).
9. R. Venugopalan, Nucl. Phys. A **590**, 147c (1995).