# A NEW CODE FOR SIMULATION OF SPACE CHARGE EFFECTS IN ISOCHRONOUS CYCLOTRONS

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The paper describes a new beam dynamics code for simulation of space charge effects in isochronous cyclotrons. The program assumes that the beam consists of many point macro-particles. The code uses a set of simplified equations of motion. It adopts continuous and slow acceleration approximation that allows one to separate betatron motion from slower energy-phase motion. Thus, the particle radial displacement r can be described as r=R+x where R is the slowly changing part due to energy-phase motion and x is the betatron part. To further simplify the problem we use the smooth focusing approximation: Azimuthally dependent focusing is substituted by its average value calculated over one turn. Thus, focusing depends on only particle s energy and can be expressed via betatron frequencies $\nu_r$, $\nu_z$. Besides, R is substituted by its average value. Such parameters of a trajectory as R, $\nu_r$, $\nu_z$, and particle rotation frequency are found from a pre-calculated table of the equilibrium orbit parameters. The table includes equilibrium orbit parameters calculated for a coasting beam with zero current for different energies. The table is prepared using measured or calculated cyclotron median plane magnetic field.

The equations of motion are (in the CGS unit system):

$$\frac{dx}{d\tau} = \frac{\tilde{p}_x}{\gamma} \quad \textbf{(1),}$$

$$\frac{d\tilde{p}_x}{d\tau} = -\gamma \frac{\nu_r^2 x}{(1+\Delta)^2} + a \frac{E_r}{\gamma^2 B_0} \quad \textbf{(2),}$$

$$\frac{dz}{d\tau} = \frac{\tilde{p}_z}{\gamma} \quad \textbf{(3),}$$

$$\frac{d\tilde{p}_z}{d\tau} = -\gamma \frac{\nu_x^2 z}{(1+\Delta)^2} + a \frac{E_z}{\gamma^2 B_0} \quad \textbf{(4),}$$

$$\frac{d\phi}{d\tau} = \frac{h}{1+\Delta}\left[(1+\varepsilon)\Delta + \varepsilon\right] - \frac{q}{2\pi(1+\Delta)}\left(\frac{dV}{dE}\right)\sin(\phi) + \frac{q}{6\pi(1+\Delta)}\left(\frac{dV_3}{dE}\right)\sin(3(\phi - \phi_{13})) + \frac{h}{(1+\Delta)}\frac{x}{R} \quad \textbf{(5),}$$

$$\frac{dE}{d\tau} = \frac{q}{2\pi(1+\Delta)}(V\cos(\phi) - V_3 \cos(3(\phi - \phi_{13}))) + \frac{qR\,E_\theta}{1+\Delta} + \frac{q\,E_r \tilde{p}_x}{\gamma} \quad \textbf{(6)}$$

where x, z are horizontal and vertical displacements from the average radius and $\tilde{p}_{x,z}=p_{x,z}/(m\omega_0)$ are the corresponding momenta. E is the energy, and $\phi$ is the phase of a particle: $\phi=\omega_{RF}\cdot t-h\cdot\theta$, where $\theta$ is the azimuth of the particle. $\tau$ is the dimensionless time, $\omega_0 t$, where $\omega_0$ is the isochronous cyclotron frequency and t is time. $B_0$ is magnetic field that corresponds to the isochronous frequency $\omega_0$. $a=c/\omega_0$ is the Cyclotron Unit Radius. *E* is electric field due to space charge in the laboratory frame, q is Q/A, V is the RF accelerating voltage, $V_3$ is the RF flat-topping voltage, $\varphi_{13}$ is the phase shift between the main accelerating voltage and the flat-topping voltage, h is the RF harmonic number. $\Delta$ describes how cyclotron field is different from isochronous. Rotation frequency can expressed via $\Delta$ as $\omega=\omega_0/(1+\Delta)$. $\varepsilon$ is the RF frequency shift and can be expressed by the formula: $\omega_{RF}=h(1+\varepsilon)\omega_0$. $\gamma$ is the relativistic factor. Values of R, $\nu_r$, $\nu_z$, V, and $\Delta$ are functions of only energy and found from the pre-calculated Equilibrium Orbit code output table.

Equations (1)-(4) describe the betatron motion. It is easy to see that these equations include the effect of adiabatic damping, which changes the transverse beam size if beam energy and/or betatron frequency change.

Equations (5) and (6) describe the energy-phase motion and are loosely based on the equations presented in [1]. Equation (5) describes the phase motion. The first term describes a phase slip due to field imperfections and RF frequency shift. The second and the third terms describe the phase compression due to magnetic field in accelerating gaps. This effect is especially important in machines with single-gap, pill-box-like cavities because accelerating field of such cavities strongly depends on radius (particle energy). The last term is proportional to x and provides radial-azimuthal coupling. This term is responsible for the very well known effect: A particle, which is involved in radial betatron oscillations, moves on a circle respectively to a particle, which has no oscillations. Though this term couples fast betatron motion with slower energy-phase motion, it is easy to see that radial betatron oscillations do not change average phase of the particle. This term becomes important in the case of high beam current. The radial component of the space charge force can change the average value of the radial displacement x from zero. As follows from (5), this leads to systematic phase slip. Equation (6) determines the rate of energy change. The first and the second terms describe the energy gain due to the main accelerating RF voltage and the flat-topping RF voltage respectively. The third and the fourth terms describe energy change due to electric field of the beam.

The classical, explicit, $4^{th}$ order Runge-Kutta method is used to solve the system of equations. The dimensionless time is the independent variable. The electric space charge field is calculated every RK sub-step, that is, the field is calculated four times per a complete RK step. Though this procedure is time consuming, it gives a self-consistent solution.

In calculating the space charge force we neglect the curvature of the bunch trajectory and straighten out the bunch. Namely, instead of the cylindrical coordinate system $(R+x,\theta,z)$ we use the Cartesian coordinate system where the radius vector of a particle is defined as $(r+x,-(R+x)\phi/h,z)$. This approximation facilitates calculation of the space charge force at high energy when relativistic effects become important (see a discussion below). The approximation is valid if at least one of the two following conditions is true: (1) the azimuthal bunch length or (2) the vertical gap is much smaller than the radius of the trajectory.

The Particle-Mesh method is used for space charge field calculation. Each time step when the field has to be calculated the charge of macro-particles is assigned to nodes of a 3D rectangular mesh. The charge of each macro-particle is distributed over 27 closest nodes. The Poisson equation is then solved on the mesh by means of the Fast Fourier Transformation [2].

We assume that the beam is confined between two horizontal, flat, grounded plates. These plates represent the vertical aperture of a cyclotron. Vertical size of the mesh equals to the gap between the plates.
The Dirichlet (zero potential) boundary condition is applied to vertical boundaries of the mesh. This boundary condition automatically takes into account all the image charges.

The periodical boundary condition is imposed in radial direction. By this we take advantage of the fact that neighboring turns are alike. The farther away bunches are, the less similar they are. However, the effect of distant bunches is very small because of image charges. Thus, the assumption of periodicity is more or less correct if the bunch shape does not change much at a distance of the order of the vertical gap. If this condition is correct, the periodic solution is then similar to self-consistent. Width of the mesh in radial direction, which equals to the length of periodicity, can be equal to either radius gain in case of accelerated beam or any arbitrary value in case of a coasting beam. Note that the periodic boundary condition allows the program to calculate the field of overlapping neighboring turns when radius gain is smaller than the radial bunch size.

Azimuthally, an isolated solution is simulated. A method of numerical solution of this problem can be found in [2]. Though correct, the method was found to be less effective in case of long bunches when the length of a bunch is comparable or larger than the vertical gap. The alternative is to impose Dirichlet boundary condition and move the boundary far enough from the edges of the bunch. It was found from numerical simulations that this distance should be approximately equal to a double vertical gap. This technique was found to work well unless the bunch is much shorter than the vertical gap. In the majority of high current cyclotrons bunch length is larger than the vertical gap. Therefore, the second technique was chosen for the Poisson solver.

Simple calculations show that the force between charged particles depends on energy. Yet, exact calculation of relativistic electromagnetic field of the beam consisting of many bunches moving with the centrifugal acceleration is difficult. The current version of the field solver assumes that the beam moves along a straight line. The rest frame of the bunch is then inertial that dramatically simplifies calculation of the field. The field is calculated in the bunch rest frame. The bunch is elongated by factor of $\gamma$ and is at rest in this frame. The field of this elongated bunch is calculated according to the discussed before procedure and then transformed to the laboratory frame according to the Lorentz transformations. Magnetic field due to space charge also appears in the laboratory frame: $\dot{\mathbf{B}}_{SC} = \dot{\beta} \times \dot{\boldsymbol{E}}_{\boldsymbol{SC}}$. The Lorentz force effectively decreases the transverse electric force that is reflected by the factor $1/\gamma^2$ in equations (2) and (4).

After the electric space charge field of the bunch has been calculated the program solves the set of equations (1)-(6) and propagates coordinates of particles. On the next step, the field of the new charge distribution is calculated and this cycle repeats until a stop-running condition is met (it can be either number of step or final average energy of the bunch). If acceleration is present, the bunch propagates outward and its energy grows. It also expands azimuthally and its radius gain normally decreases. To take into account all these effects and obtain a self-consistent solution length of the mesh, its position, and width are recalculated every step. As was mentioned above, width of the mesh is kept equal to the radius gain.

*Test examples:*

*a. Uniformly charged disk*

Motion of a coasting beam shaped as a uniformly charged, thin disk was simulated in the ideal, azimuthally symmetric, isochronous magnetic field. $10^5$ macro-particles were initially distributed in the median plane over a circular area with radius 2 cm. Each particle was distributed over the area randomly with the uniform probability. Particles initially had no betatron oscillations. The magnetic field was 3.3 kGauss, average energy was 5 MeV, and the bunch current was 1 mA. The radial width and the height of the mesh were 10 cm. The length was 44 cm.

Figure 1 shows trajectories of six particles in the disk reference frame. The particles have been tracked for 52 turns. They moved on almost circular trajectories inside the disk. The inner particles rotated slower than outer ones. The period of the rotation was calculated from the figure 1 and compared with the theoretically calculated period for an isolated, uniformly charged disk distribution. Electric field inside the isolated disk distribution is only radial (respectively to the disk). The azimuthal average, drift velocity of the particles inside the disk is described by the formula: $\mathrm{v}_\theta = \mathrm{c}\, Er / \mathrm{Bz}$. Using this formula, one can easily estimate the period of rotation of a particle inside the disk distribution as $\mathrm{n} = \mathrm{r_p B_z} / (\mathrm{a}\, E_r)$. That is, the disk makes n turns in the magnetic field $B_z$ while a particle with the displacement $r_p$ from the center of the disk makes one turn in the disk reference frame under the influence of the space charge field $E_r$. a is the Unit Cyclotron Radius. The field $E_r$ can be calculated, for example, numerically, evaluating the expression:

$$E_r(r) = 2\sigma \frac{\partial}{\partial r} \int_0^{R_{disc}} \int_0^{\pi} \frac{R\,dR\,d\theta}{\sqrt{r^2 + R^2 - 2rR\cos(\theta)}}$$

where $\sigma$ is the surface charge density and $R_{disc}$ is the radius of the disk. Figure 2 shows the period of the rotation calculated from figure 1 and estimated theoretically vs. particle s radius respectively to the center of the disk. Both results are in more or less good agreement except the edge of the disk. As was mentioned before, the field solver works efficiently if the longitudinal beam size is larger than the vertical mesh size. In the example, the disk diameter (2 cm) was smaller than the vertical mesh size (10 cm) in order to simulate an isolated solution in all three directions. Increasing the number of mesh nodes in longitudinal direction solves this problem but increases the amount of computations.

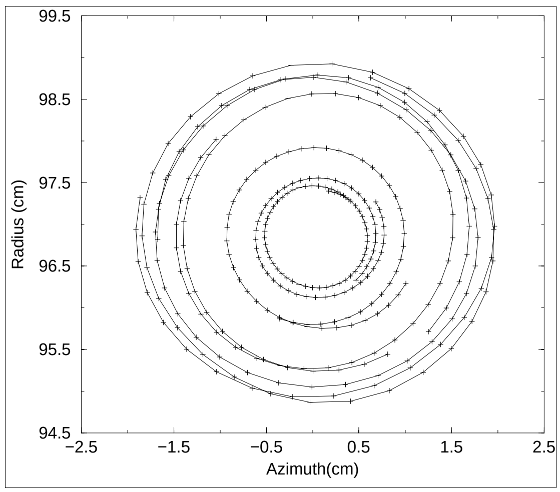


Figure 1: Simulated trajectories of six particles inside the disk-like bunch. The crosses represent consecutive turns. The particles move on almost circular trajectories. The inner particles rotate slower than outer ones because the electric field of the disk increases with radius.

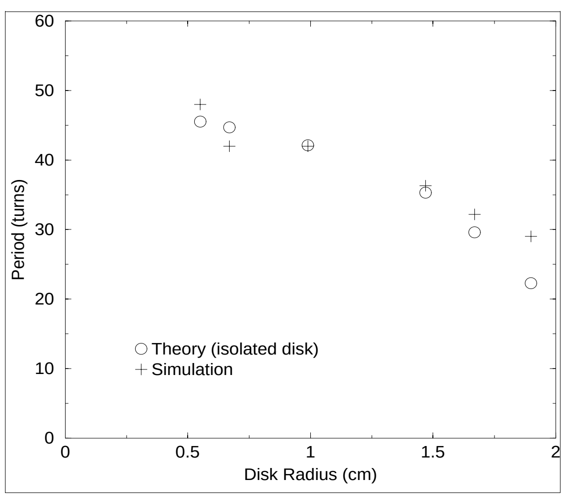


Figure 2: The period of the rotation calculated from figure 1 and theoretically vs. particle s radius respectively to the disk center.

Because particle trajectories are almost circular inside the disk, the disk shape has to stay almost unchanged. Figure 3 shows the envelope of the initial distribution and after 50 turns. One can see, that beam shape has little changed for 50 turns. Figure 4 shows the r.m.s. radius vs. turn number. As follows from the figure the radial distribution of particles also does not change much within 50 turns.

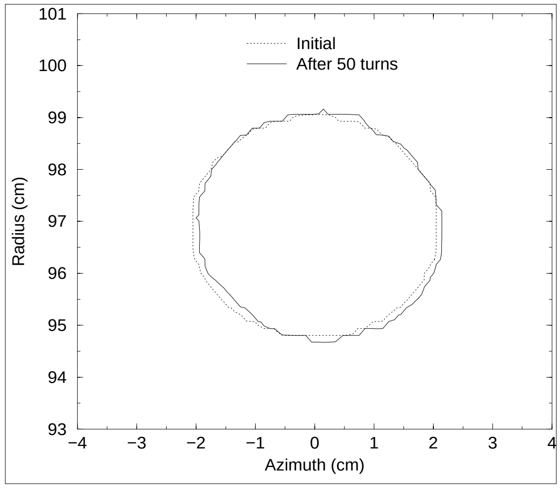


Figure 3: The envelope of the disk-like bunch: initial and after 50 turns. The bunch shape changes little because particles move on almost circular trajectories inside the bunch.

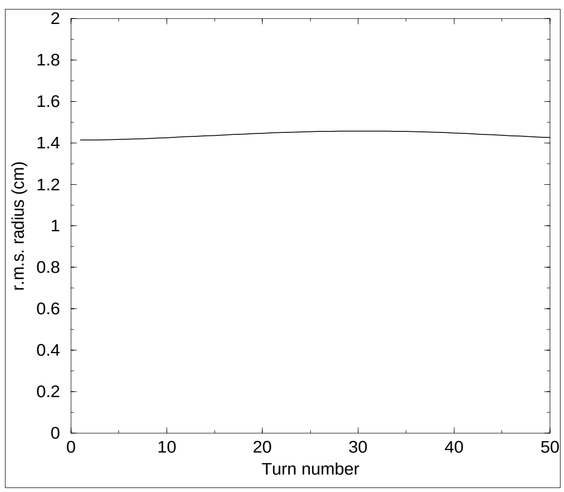


Figure 4: The r.m.s. radius of a disk-like distribution as function of turn number. From figures 3 and 4 one can conclude that distribution of particles inside the disk-like bunch changes little for 50 turns.

*b. Comparison with the program PICN [3]*

Motion of a 2 mA, 5 MeV, coasting beam in PSI Injector II was simulated. Radial and vertical sizes of the beam were 8 and 4 mm respectively. In azimuthal direction particles were randomly distributed with probability described by the Gaussian curve with r.m.s. size of 8.5 mm. All particles with azimuthal deviation larger than $4\sigma$ were disregarded. The total number of used particles was $10^5$. Figure 5 shows the result of the simulation. Similar simulation was also carried out using the program PICN [3] developed at PSI. The program PICN treats the beam as a set of vertical needles with fixed height moving in the median plane. The result of simulation using PICN is presented in figure 6. Initial distributions used in our program and in PICN were similar. Yet, there were some obvious differences between these two because of absolutely different techniques used to distribute particles. As follows from the figures, both programs gave similar beam behavior in the PSI Injector II: The space charge force caused the test bunch to deform into s-like galaxy shape in a few turns. After approximately 10 turns the bunch transformed into a round distribution, which has changed little thereafter.

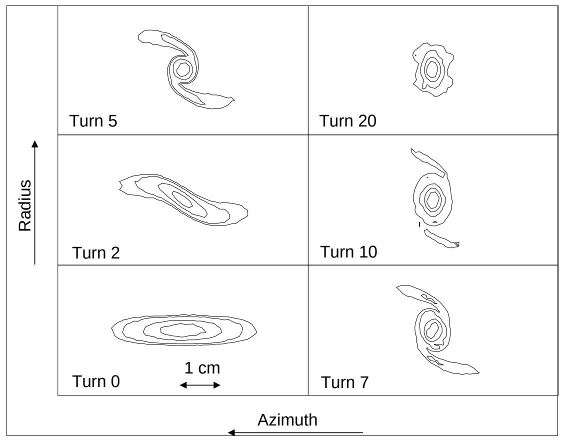


Figure 5: Simulation of motion of a coasting bunch in PSI Injector II: our program. Contour lines represent 10%, 20%, 50%, and 80% of the maximum charge density. Because of the space charge force the bunch deform into a galaxy-like shape in a few turns.

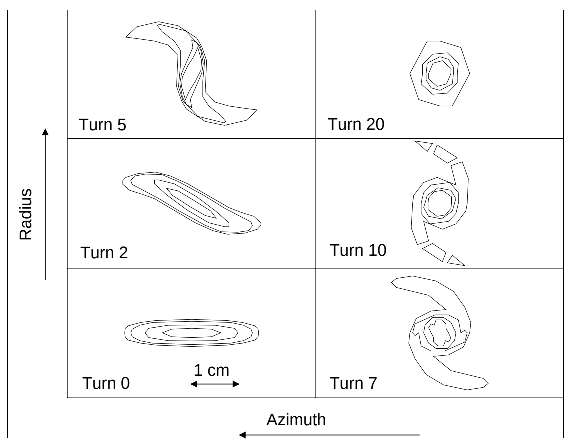


Figure 6: Simulation of motion of a coasting bunch in PSI Injector II: PICN. Contour lines represent 10%, 20%, 50%, and 80% of the maximum charge density. Both programs gave similar result: The vortex motion causes the bunch to transform into a s-like shape in a few turns. After approximately 10 turns the bunch deforms into a round distribution with changes little thereafter.

## References


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3. S. Adam. Proc. 14th. Int. conf. On Cyclotrons and their Application. Cape Town (1995), p.446